A Discrete General Class of Continuous Distributions

By

Taghreed Abdulrahman Al-Masoud

A thesis submitted for the requirements of the degree of Master of Science (Statistics)

Supervised By
Dr. Sohila Hamed Elgayar
Dr. Mahmoud Abdul Mo'men Atallah

FACULTY OF SCIENCE
KING ABDUL AZIZ UNIVERSITY
JEDDAH-SAUDI ARABIA
Jumada al-thani 1434H- May 2013G

بسم الله الرحمن الرحيم

A Discrete General Class of Continuous Distributions

By

Taghreed Abdulrahman Al-Masoud

This thesis has been approved and accepted in partial fulfillment of the requirements for the degree of Master of Science (STATISTICS)

EXAMINATION COMMITTEE

	Name	Rank	Field	Signature
Internal Examiner	Prof. Dr. Abdullah H. Al-Harbey	Professor	Statistics	
External Examiner	Dr. Mohamed Kayid Abdelgawwad	Associate Professor	Statistics	
Co-supervisor	Dr. Mahmoud Abdulmo'men Atallah	Assistant Professor	Statistics	
Supervisor	Dr. Sohaila Hamed Elgayar	Assistant Professor	Statistics	

KING ABDUL AZIZ UNIVERSITY

Jumada II 1434H - May 2013

Dedicated to

My beloved family, and

Every one supported me.

ACKNOWLEDGEMENTS

I love carried out this study for the Master of Science in Statistics (Discretizing a general class of continuous distributions) seeking the help from Allah.

I would like to extend my heartfelt thanks to my supervisor, Dr.Sohaila Elgayar, for all her concern, advice and all the precious time she has offered to supervise this work. I am also grateful for Dr. Mahmoud Atallah for his valuable support.

I also wish to convey my appreciations to my graduate study teachers, Prof. Gannat Al-Dayian, Dr. Neamat Kotb, and Dr. Samia Adham for strengthening my knowledge in Statistics. Special thanks go to Dr. Lutfiah Al-Turk, the director of the department for her support.

Special acknowledgement goes to my wonderful mother who provided a comfortable working environment for me, and who patiently tolerated the disruption made thereby to her life. Deep thanks to my father, husband, brothers and sister for their continuous support. Special thanks to my children Battar 3-years and Lateen 2-years for not giving them the necessary time and full care and devotion.

Last but not least, I am also thankful to those, who have directly or indirectly, helped me to be better in life.

"Let us work together to save our planet forever"

Taghreed Al-Masoud

A Discrete General Class of Continuous Distributions

Taghreed Al-Masoud

ABSTRACT

Quality and validity of products of all kinds become very important to pay attention so that they can withstand the competition in the market due to the multiplicity of forms and sources of products. It is well known that the consumer cares about the quality of industrial products of all types which are displayed in markets. They should be of high efficiency and longer life. In concordance with the requirements of the consumer, factory owners seek desperately to attract consumers to their products. The most important result of this research is the emergence of the so-called guarantee certificates that have shared preference among consumers for alternative warranty-free product. Therefore it is very necessary to shed light on how to determine the appropriate duration of guarantee certificates accurately, otherwise the error identified could cost companies huge losses. Determining of the appropriate duration of such certificates requires the collection of information about the product through the design of the so-called life-testing experiments or tests of Reliability. This should be done before sending product to markets because the information obtained from such experiments – in addition to their importance in determining the duration of guarantee certificates - can be used in other fields. For instance, in pharmaceutical studies, we would like to design life-testing experiments on drugs to determine their effectiveness duration and expiry date. Indeed, there are so many areas where designing such experiments is of paramount importance.

In life-testing experiments, sometimes it becomes impossible to measure the life of a product or its expiry date by continuous scale, like in turning a device on and off during its lifetime, because turning on and off is a random separate variable and in some cases validity of the data is measured by the number of operating times. As for survival analysis, it is possible to record the number of days remaining for lung cancer patients

During the treatment period. In this context, standard discrete distributions like geometric and negative binomial have been employed to model life time data.

In this thesis a general class of continuous distributions is considered. Furthermore, a generated discrete life distribution based on a continuous distribution, by using the general approach of discretizing a continuous distribution.

Several discrete lifetime distributions are proposed with their properties and some measures of reliability, such as discrete modified Weibull extended, discrete modified Weibull type I, discrete modified Weibull type II, discrete Chen (2000), and discrete linear failure rate distributions.

A Mathcad simulation study is conducted to the properties and the distributional characteristics of the new discrete distributions. The performance of the estimators of the parameters is presented.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS AND TERMINOLOGY

sf	Survival function.
SRF	Second rate of failure.
IFR	Increasing failure rate.
IFRA	Increasing failure rate in average.
mgf	moment generating function
pgf	probability generating function
PM	Proportion method.
MM	Moments method.
ML	Maximum likelihood method.
cdf	cumulative distribution function.
pmf	Probability mass function.
MWE	Modified Weibull extension.
MW(I)	Modified Weibull type one.
MW(II)	Modified Weibull type two.
DMWE	Discrete Modified Weibull extension.
DMW(I)	Discrete modified Weibull type one.
DMW(II)	Discrete modified Weibull type two.
DChen	Discrete Chen.
DLFR	Discrete Linear failure rate.
DEXV	Discrete Extreme value.

Chapter I

Introduction

1.1 Foreword

Survival analysis is a branch of Statistics which deals with death in biological organisms and failure in mechanical systems. This topic is called Reliability theory or reliability analysis in engineering and it is called duration analysis or duration modeling in Economics or Sociology. More generally, survival analysis involves the modeling of time to event data. In this context, death or failure is considered an "event" in the survival analysis literature.

1.2 Research Gaps

In life testing experiments, it is sometimes impossible or inconvenient to measure the life length of a device, on a continuous scale. For example, in the case of an on/off switching device, the life time of the switch is a discrete random variable. In many particular situations, reliability data are measured in terms of the number of runs, cycles, or shocks the device sustains before it fails. In survival analysis, it may record the number of days of survival for lung cancer patients since therapy, or the times from remission to relapse are also usually recorded in number of days. Many continuous distributions can be discretized. In this context, the geometric and negative binomial distributions are known discrete alternatives for the exponential and gamma distributions, respectively.

1.3 Motivation

Discrete distributions are finding their way into survival analysis. The lifetimes of many components are being measured by the number of completed cycles of operation or strokes. Even for a continuous operation, involving a continuous measurement of lifetime, observations made at periodic time points give rise to a discrete situation, and a discrete model may be more appropriate. Nakagawa and Osaki (1975) discretized the Weibull distribution. Nakagawa (1978) defined the discrete extreme distributions. Stein and Dattero (1984) discussed a new discrete Weibull distribution. Roy (2004) proposed a discrete Rayleigh distribution. Krishnah and Pundir (2009) presented the discrete Burr XII and Pareto distributions. Jazi, Lia and Alamatsaz (2010) proposed the discrete inverse Weibull distribution. The discrete version of Lindley distribution was introduced by Deniz et al. (2011). Al-Dayian and Al-Huniti (2012) introduced the discrete Burr Type III distribution.

The modified Weibull extension distribution was proposed by Xie *et al.* (2002). It is an extension of a two parameter model proposed by Chen (2000), and it involves three parameters. This model is capable of modeling bathtub-shaped failure rate lifetime data. It can be written as an exact form of a mixture of distributions under certain conditions, and provides extra flexibility to the density function over positive integer.

The modified Weibull distribution has been introduced by Sarhan and Zaindin (2009a). This distribution generalizes the exponential, Rayleigh, linear failure rate, and Weibull distributions. These are the most commonly used distributions in reliability and life testing. They have several desirable properties and nice physical interpretations. The modified Weibull distribution that generalizes all the above distributions can be used to describe several reliability models.

A new lifetime distribution capable of modeling a bathtub-shaped hazard-rate function which called new modified Weibull distribution studied by Lai *et al*. (2003). It can be considered as a useful three-parameter generalization of the Weibull distribution.

1.4 Methodology

Discrete distributions are used in reliability when lifetime measurements are taken in a discrete manner. Many continuous distributions can be discretized. There exist two approaches of discretizing distributions. The first approach of discretizing reliability distributions has been defined by Nakagawa and Osaki (1975). This approach has been used in the present study.

1.5 Contributions

In our present study a general class of continuous distributions is discretized. Some generalized discrete models such as DMWE, DMW (I) and DMW (II) distributions are introduced. Some reliability measures and characteristics of the discretized general class are investigated. The parameters of the studied distributions are estimated. Three estimation methods are used. The used estimation methods are: the proportion method (PM), the method of moments (MM), and the maximum likelihood method (MLM). The estimation results are compared. New discretized distributions are obtained. This includes discrete Chen distribution (DChen) and discrete linear failure rate distribution (DLFR). Simulation studies using MathCAD software are conducted. Theoretical and numerical results are obtained.

Chapter II

Definitions and Notations

This chapter involves some definitions, which used throughout the thesis.

2.1 Reliability Measures

The basic definitions of reliability measures for systems, with continuous and discrete lifetimes, are given.

2.1.1 Reliability Measures in the Continuous Case

Let T be random lifetime with a continuous distribution onR^{+}.

Definition 2.1: The reliability function$S(x)$ is defined for all $t \geq 0$ as follows

$S(t) = P(T > t)$.

Definition 2.2: The failure rate function$r(t)$ is defined for all $t \geq 0$ as follows

$r(t) = f(t)/S(t)$.

Definition 2.3: The residual reliability function $R(s|t)$ is defined for all $, s \geq 0$, as

$R(s|t) = P(T > s + t|T > t)$.

$= S(s + t)/S(t)$.

Definition 2.4: The cumulative hazard function $H(t)$ is defined for all $t \geq 0$ by $H(t) = -lnS(t)$.

Definition 2.5: T is said to be increasing failure rate (IFR) if and only if, equivalently:

IFR1: The failure rate function $r(t)$ is an increasing function of t.

IFR2: For all$s \geq 0$, the residual reliability function $R(s|t)$ is decreasing with t.

Definition 2.6: T is called increasing failure rate in average (IFRA) if and only if, equivalently:

IFRA1: $[S(t)]^{1/t}$ is a decreasing function of t.

IFRA2: $H(t)/t$ is an increasing function of t.

The equivalence between IFRA1 and IFRA2 is immediate since $H(t) = -\ln S(t)$.

(Barlow (2001))

2.1.2 Reliability Measures in the Discrete Case

Let the random variable T be a discrete system lifetime. T is defined over the set of positive integers N^*. Let $p(k) = P(T = k)$ be the probability that the system fails at time k.

Definition 2.7: The discrete reliability function $S(k)$ is defined to be the probability that the system is still alive at time k. That is

$$S(k) = P(T > k)$$

$$= \sum_{i=k+1}^{\infty} p(i) \quad (2.1)$$

Definition 2.8: The discrete failure rate function $r(k)$ is defined, for all $k \in N^*$, by

$$r(k) = P(T = k|T \geq k)$$

$$= \frac{S(k-1)-S(k)}{S(k-1)} \quad (2.2)$$

Definition 2.9: The discrete cumulative hazard function $H(k)$ is defined, for all $k \in N^*$, by

$$H(k) = \sum_{i=1}^{k} r(i) \quad (2.3)$$

Definition 2.10: The discrete residual reliability function at time k is denoted by $R(i|k)$ and is defined for all $i \in N^*$, by

$$R(i|k) = P(T > k + i|T > k)$$

$$= \frac{S(k+i)}{S(k)} \quad (2.4)$$

Definition 2.11: The second rate of failure sequence$\{SRF(k)\}_{k\geq 1}$, is defined as

$$SRF(k) = -lnS(k) + lnS(k-1)$$

$$= ln\frac{S(k-1)}{S(k)} \quad (2.5)$$

Definition 2.12: T is said to be a discrete increasing failure rate (IFR) if and only if, equivalently:

IFR1: $\{r(k)\}_{k\geq 1}$ is an increasing sequence.

IFR2: For all $s \in N^*$, $\{R(s|k)\}_{k\geq 1}$ is a decreasing sequence.

IFR3: $\{\ln S\,(k)\}_{k\geq 1}$ is a concave sequence.

Definition 2.13: T is said to be a discrete increasing failure rate in average (IFRA) if and only if, equivalently:

IFRA1: $\left\{(S(k))^{1/k}\right\}_{k\geq 1}$ is a decreasing sequence.

IFRA2: $\{H(t)/k\}_{k\geq 1}$ is an increasing sequence. (2.6)

(Barlow (2001))

2.2 Some Methods of Estimation

The maximum likelihood, moments, proportion methods of estimation will be discussed. These methods are applied to estimate the unknown parameters of the considered probability distributions.

2.2.1 The Proportion Method

The proportion method (PM) proposed by Khan *et al*. (1989) is used to estimate the parameters. Let $x_1, x_2, \ldots, x_k$ be an observed sample from a distribution with probability mass function $P(x; \theta_1, \ldots, \theta_k)$. Define the indicator function $I_u(x_i)$ of the value u by

$$I_u(x_i) = \begin{cases} 1 & if\ x_i = u, \\ 0 & if\ x_i \neq u. \end{cases}$$

Denote by $f_u = \sum I_u(x_i)$ the frequency of the value u in the observed sample. Therefore, the proportion (relative frequency) $r_u = f_u/n$ is can be used to estimate the probability $P(u; \theta_1, \ldots, \theta_k)$.

Consequently, the probability $P(1; \theta_1, \ldots, \theta_k)$ is the proportion r_1 in the observed sample. Therefore, $\hat{\theta}_1$ is the estimate of θ_1, with f_1 as the observed frequency of the value 1.

Similarly, the probability $P(2;\theta_1,\dots,\theta_k)$ is the proportion r_2 in the observed sample. Therefore, $\hat{\theta}_2$ is the estimate of θ_2 with f_2 as the observed frequency of the value 2, and so on. (Khan et al. (1989))

2.2.2 The Moments Method

Consider a population with a pdf $f(x;\theta_1,\dots,\theta_k)$, depending on one or more parameters $\theta_1,\dots,\theta_k$. The j^{th} moment about the origin μ_j', is defined by

$$\mu_j'(\theta_1,\dots,\theta_k) = E(X^j) \qquad j = 1,2,\dots,$$

Let $X_1,\dots,X_n$ is a random sample of size n from f$(x;\theta_1,\dots,\theta_k)$. The j^{th} sample moment M_j' is defined by

$$M_j' = \frac{1}{n}\sum_{i=1}^{n} x_i^j,\ j = 1,2,\dots..$$

The method of moments is to choose as estimators of the parameters $\theta_1,\dots,\theta_k$ the values $\hat{\theta}_1,\dots,\hat{\theta}_k$ that render the population moments equal to the sample moments. In other words, the values $\hat{\theta}_1,\dots,\hat{\theta}_k$ are solutions of the following k equations

$$M_j' = \mu_j'(\theta_1,\dots,\theta_k),\ j = 1,2,\dots.,k$$

(Bain and Engelhardt (1992))

2.2.3 The Maximum Likelihood Method

The likelihood function of the n random variables $X_1,\dots,X_n$ is defined to be the joint density of n random variables, say$f_{X_1,\dots,X_n}(x_1,\dots,x_n;\theta_1,\dots,\theta_k)$, which is considered to be a function of $\theta_1,\dots,\theta_k$. In particular, if $X_1,\dots,X_n$ is a random sample from the density $f(x;\theta_1,\dots,\theta_k)$, then the likelihood function L is defined as

$L = L(\theta_1, \theta_2, \dots, \theta_k) = \prod_{i=1}^{k} f(x_i; \theta_1, \dots, \theta_k)$

The point at which the likelihood function L (or the log likelihood function $\ln L$) is a maximum, is the solution of the following either system of k equations:

(S1) $\frac{\partial L(\theta_1,\theta_2,\dots,\theta_k)}{\partial \theta_j} = 0,$ j=1,….,k

(S2) $\frac{\partial \ln L(\theta_1,\theta_2,\dots,\theta_k)}{\partial \theta_j} = 0,$ j=1,…,k

(Mood and Graybill (1974))

2.3 Inverse Transform Method for Simulation from a Discrete Distribution

The algorithm of simulating a sequence of the random numbers x_i of the discrete random variables X with pmf $P(X = x_i) = p_i$, $\sum_{i=0}^{m} p_i = 1$, and a cdf F(x), where *m* may be finite or infinite can be described as

Step 1: Generate a random number *u* from uniform distribution *U* (0, 1).

Step 2: Generate random number x_i based on

$if u < p_0 = F(x_0),$ then $X = x_0,$

$if p_0 < u \leq p_0 + p_1 = F(x_1),$ then $X = x_1,$

$\vdots$

$if \sum_{j=0}^{m-1} p_j < u < \sum_{j=0}^{m} p_j = F(x_m),$ then $X = x_m,$

To generate n random numbers $x_1, \dots, x_n$ repeat **Step 1** and **Step 2** *n* times.

Chapter III

Literature Review

3.1 Introduction

In reliability theory, many suggested continuous lifetime models are studied. However, it is sometimes impossible or inconvenient to measure the life length of a device, on a continuous scale. When the grouped lifetimes of individuals in some populations refers to an integral number of cycles of some sort, it may be desirable to treat it as a discrete random variable. Fortunately, many continuous distributions are discretized.

A new model, called an Extended Weibull or Modified Weibull extension (MWE) distribution is useful for modeling this type of failure rate function. This distribution is easy to use while it can achieve even higher accuracy compared with other models. Hence, the Extended Weibull serves is a good alternative distribution when, needed models have bathtub-shaped failure rate.

The modified Weibull Type I (MW (I)) distribution can be used to describe several reliability models. This distribution generalizes the Exponential, Rayleigh, linear failure rate and Weibull distributions.

The modified Weibull Type II (MW (II)) distribution will introduced as an extension of the Weibull model. This model will consider a three-parameter generalization of the Weibull distribution.

Two methods of discretizing the continuous distributions are discussed in the present chapter. The modified Weibull extension (MWE) distribution, modified Weibull Type I (MW(I)) distribution, modified Weibull Type II (MW(II)) distribution are discussed and their properties are presented.

3.2 Discretizing the Continuous Distributions

A continuous failure time model can be used to generate a discrete model by introducing a grouping on the time axis. Two methods of discretizing the continuous distributions will be explained in this section.

3.2.1 First Discretizing Method

If the underlying continuous failure time X has the survival function (sf) $S(x) = P[X \geq x]$ and time are grouped into unit interval so that the discrete observed variable is $dX = [X]$, where $[X]$ denotes the largest integer part of X, the probability mass function of dX can be written as

$$
\begin{aligned}
P(x) &= P[dX = x] \\
&= P[x \leq dX < x + 1] \\
&= S(x) - S(x + 1); \quad x = 0,1,2, \dots ..
\end{aligned}
$$

The probability mass function of dX can be viewed as a discrete concentration of the probability density function of X.

Nakagawa and Osaki (1975) are first to use this approach. They discretized the Weibull distribution with two parameters and studied the properties of the discrete Weibull distribution such as the failure rate. Nakagawa (1978) defined the discrete extreme distributions. The application to an *n*-unit parallel system in random environment was shown. After that Stein and Dattero (1984) discussed a new discrete Weibull distribution and compared it with the discrete Weibull distribution introduced by Nakagawa and Osaki (1975). They proved that the hazard rate of the discrete Weibull distribution is similar to that of the continuous Weibull. They also proved that the exact lifetime distribution of a specific system and the lifetime converge to that given by the continuous Weibull thus showing the connection between the two distribution. Khan et al. (1989) discussed the two discrete Weibull distributions that were introduced by Nakagawa and Osaki (1975), and Stein and Dattero (1984). They presented the so-called proportion method to estimate the parameters. Dilip Roy (2004) proposed a discrete Rayleigh distribution. He deliberated on the problem of discretization of the Rayleigh distribution, to retain resemblance with its continuous counterpart, and used the corresponding properties of the continuous Rayleigh distribution. He studied the estimated problem of the underlying parameter. Burr XII and Pareto distributions were considered as a continuous lifetime model and their discrete analogues with their distributional properties and reliability characteristics derived by Krishnah and Pundir (2009). They discussed the maximum likelihood estimation in discrete Burr (DB (XII)) distribution and discrete Pareto (DP) distribution in detail with simulation study. Jazi, Lia and Alamatsaz (2010) proposed and studied an analogue of the continuous inverse Weibull distribution. They presented four methods for estimating the parameters of the discrete inverse Weibull distribution. The discrete version of

Lindley distribution was introduced by Deniz et al. (2011), by discretizing the continuous failure model of the Lindley distribution. Also, a closed form compound discrete Lindley distribution is obtained after revising some of its properties. Finally, Al-Dayian and Al-Huniti (2012) introduced the discrete Burr type III distribution as a suitable lifetime model and developed its distributional characteristics. The maximum likelihood and Bayes estimations are illustrated.

3.2.2 Second Discretizing Method

For any continuous random variable X on R with pdf $f(x)$, one can define a discrete random variable Y that has integer support on $(-\infty, \infty)$ as follows

$$P(Y = k) = \frac{f(k)}{\sum_{j=-\infty}^{\infty} f(j)}; \qquad k = 0, \pm 1, \pm 2, \ldots$$

Kemp (1997) used this method to obtain a discrete analogue of the normal distribution as the one that is characterized by maximum entropy, specified mean and variance, and integer support on $(-\infty, \infty)$. Inusha and kozubowski (2006) derived a discrete version of the Laplace distribution. They presented various representations of discrete Laplace variables and discussed its properties. The maximum likelihood and the method-of-moments estimators are obtained and their asymptotic properties are established.

3.3 Modified Weibull Extension Distribution

Models with bathtub-shaped failure rate function are useful in reliability analysis, and particularly in reliability-related decision-making and cost analysis. A modified Weibull extension (MWE(α, β, λ)) model is useful for modeling this type of failure rate function. It can be a generalization of the Weibull distribution.

Xie *et al*. (2002) proposed and discussed an extended new distribution (MWE(α, β, λ)) capable of modeling bathtub-shaped failure-rate lifetime data. This model can be a generalization of the Weibull distribution and it is very flexible. This new model only contains three parameters and it is related to exponential and Weibull distributions in an asymptotic manner.

Nadarajah (2005) derived explicit algebraic formulas for k^{th} moment of the distribution. The cumulative distribution function of the MWE (α, β, λ) distribution is given by

$$F(t) = 1 - exp\left[\lambda\alpha\left(1 - exp\left(\frac{t}{\alpha}\right)^{\beta}\right)\right]; \quad t > 0, \lambda > 0, \alpha > 0, \beta > 0.$$

The corresponding probability density function has the form

$$f(t; \lambda, \alpha, \beta) = \lambda\beta\left(\frac{t}{\alpha}\right)^{\beta} exp\left[\left(\frac{t}{\alpha}\right)^{\beta} + \lambda\alpha\left(1 - exp\left(\frac{t}{\alpha}\right)^{\beta}\right)\right]; t > 0, \lambda > 0, \alpha, \beta > 0.$$

The reliability function is

$$R(t) = exp\left[\lambda\alpha\left(1 - exp\left(\frac{t}{\alpha}\right)^{\beta}\right)\right].$$

The corresponding failure rate has the following form

$$r(t) = \lambda\beta\left(\frac{t}{\alpha}\right)^{\beta-1} exp\left[\left(\frac{t}{\alpha}\right)^{\beta}\right].$$

Xie *et al*. (2002) studied the shape of the failure rate function and deduced that when $\beta \geq 1$, the failure rate function is an increasing function and is a bathtub-shaped function when $\beta < 1$.

The mean time-to-failure of the distribution is

$$E(T) = \int_0^{+\infty} exp\left[\lambda\alpha\left(1 - exp\left(\frac{t}{\alpha}\right)^{\beta}\right)\right] dt.$$

The above integral is difficult to calculate analytically. Hence, numerical integration is usually needed.

The variance of the time-to-failure is

$$Var(T) = 2\int_0^{+\infty} t\ exp\left[\lambda\alpha\left(1 - exp\left(\frac{t}{\alpha}\right)^{\beta}\right)\right]dt - \mu^2.$$

This expression has to compute numerically.

The MWE (α, β, λ) distribution related to Weibull distribution. When $\alpha = 1$, the MWE (α, β, λ) distribution reduces to the model by Chen (2000). He proposed this model with bathtub shape or increasing failure rate function and discussed the exact confidence intervals and exact joint confidence regions for the parameters based on type II censored samples. Weibull distribution is an asymptotic case of the MWE(α, β, λ) distribution. This occurs when the scale parameter α becomes very large or approaches infinity while $\lambda^{-1}\alpha^{\beta-1}$ remains constant. In this case, the MWE (α, β, λ) distribution becomes a standard two-parameter Weibull distribution. It will be capable in handing both decreasing and increasing failure rate. This in fact is a special case of bathtub curve. A further special case is, when $\beta = 1, \alpha$ is large enough and $\lambda^{-1}\alpha^{\beta-1}$ is a constant, the MWE(α, β, λ) distribution reduces to the exponential distribution with parameter $\lambda^{-1}\alpha^{\beta-1}$.

Parameter estimation is usually a difficult problem as even for two- parameter Weibull distribution. Methods like maximum likelihood estimation will not yield a closed form solution. Different estimation methods are used.

Xie *et al*. (2002) also estimated the distribution graphically. Simple graphical estimates are obtained. When $\lambda\alpha = 1$, the model is simplified to

$$R(t) = exp\left[1 - \left(\frac{t}{\alpha}\right)^{\beta}\right].$$

For the estimation of the parameters a graphical method is developed. A similar transformation to the Weibull transformation is

$x = \ln t,$

$y = \ln\{\ln(1 - \ln R(t))\}.$

If the life time data follows this model with $\lambda\alpha = 1$, then the plot of y versus x can be fitted with a straight line. Furthermore, β is the slope of the regression line. The estimation of α is obtained from the y-interception, and $\hat{\lambda} = 1/\hat{\alpha}$. The line is

$y = \beta x - \beta \ln \alpha, -\infty < x < \infty.$

The three-parameter MWE(α, β, λ) distribution is the general case. The traditional Weibull plot does not yield a straight line. When t is small the first part of the data on the Weibull plot is considered and can be observed as an approximate estimation of the parameters.

With the transformation $y = ln(1 - ln\overline{R(t)})$ and $x = lnt,$ a line can be obtained when plotting y versus x which satisfies the equation:

$y = \beta x + ln\left(\lambda\alpha^{1-\beta}\right).$

The slope of the regression line estimates the parameter β. The y- intercept equals $ln\left(\lambda\alpha^{1-\beta}\right).$

When t is large, the Weibull transformation

$$ln[- ln\, R(t)] = \ln(\lambda\alpha) + \ln\left[e^{\left(\frac{t}{\alpha}\right)^{\beta}} - 1\right].$$

For the second term, when t is large

$$\ln\left\{\left[e^{\left(\frac{t}{\alpha}\right)^{\beta}}-1\right]\right\}=\ln\left\{\left[1-e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]\right\}-\ln\{e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\}$$

$$=-e^{-\left(\frac{t}{\alpha}\right)^{\beta}}+\left(\frac{t}{\alpha}\right)^{\beta}.$$

Since, when t is large, the first term approaches zero, and the asymptotic curve is $(t/\alpha)^{\beta}$in this case. Hence, by taking another log, a straight line for large t can be used and graphical estimates can be obtained.

Xie *et al*. (2002) derived the maximum likelihood estimators of the parameters of the MWE(α,β,λ) distribution. Let $t_1,\ldots\ldots,t_k$ are the time-to-failure of the k failed components from a sample consisting of n components under type II censoring.

The underling likelihood function is:

$$L=\lambda^k\beta^k\prod_{i=1}^{k}\left(\frac{t_i}{\alpha}\right)^{\beta-1}\exp\{\sum_{i=1}^{k}\left(\frac{t_i}{\alpha}\right)^{\beta}+\sum_{i=1}^{k}\lambda\alpha\left(1-e^{\left(\frac{t_i}{\alpha}\right)^{\beta}}\right)$$

$$+(n-k)\lambda\alpha\left(1-e^{\left(\frac{t_k}{\alpha}\right)^{\beta}}\right)\}.$$

The log likelihood function is

$$lnL=kln\lambda+kln\beta+n\lambda\alpha+(\beta-1)\sum_{i=1}^{k}ln\frac{t_i}{\alpha}+\sum_{i=1}^{k}\left(\frac{t_i}{\alpha}\right)^{\beta}$$

$$-\lambda\alpha\sum_{i=1}^{k}exp\left[\left(\frac{t_i}{\alpha}\right)^{\beta}\right]-(n-k)\,\lambda\,\alpha\,exp\left[\left(\frac{t_k}{\alpha}\right)^{\beta}\right].$$

Equating to zero the first derivative of the log likelihood function with respect to λ, where α and β are assumed known, the maximum likelihood estimator of the parameter λ can be obtained in the form

$$\hat{\lambda} = k\left(\alpha\sum_{i=1}^{k}\left(\frac{t_i}{\alpha}\right)^{\beta} + (n-k)\alpha e^{\left(\frac{t_k}{\alpha}\right)^{\beta}} - n\alpha\right)^{-1}.$$

Finally, by taking the partial derivative with respect to β *and* α, when λ is assumed known, the following two equations follow

$$\frac{\partial lnL}{\partial\beta} = \frac{k}{\beta} + \sum_{i=1}^{k} ln\frac{t_i}{\alpha} + \sum_{i=1}^{k}\left(\left(\frac{t_i}{\alpha}\right)^{\beta} ln\frac{t_i}{\alpha}\right) - \lambda\alpha\sum_{i=1}^{k}\left(exp\left(\frac{t_i}{\alpha}\right)^{\beta}\left(\frac{t_i}{\alpha}\right)^{\beta} ln\frac{t_i}{\alpha}\right) = 0,$$

$$\frac{\partial lnL}{\partial\alpha} = \frac{k(\beta-1)}{\alpha} + n\lambda - \frac{1}{\alpha}\sum_{i=1}^{k}\left(\frac{t_i}{\alpha}\right)^{\beta} - \lambda\sum_{i=1}^{k}\left(exp\left(\frac{t_i}{\alpha}\right)^{\beta}\left(1-\left(\frac{t_i}{\alpha}\right)^{\beta}\right)\right)$$

$$-(\mathrm{n}-\mathrm{k})\lambda\mathrm{exp}\left(\frac{\mathrm{t_k}}{\alpha}\right)^{\beta}\left(1-\left(\frac{\mathrm{t_k}}{\alpha}\right)^{\beta}\right) = 0.$$

These equations are difficult to be solved analytically for β *and* α, and a suitable software package can be used to solve them numerically.

Nadarajah (2005) derived the following explicit algebraic formula for the k^{th} moment of the modified Weibull distribution

$$E(X^k) = k\int_0^{\infty} x^{k-1}R(x)dx = k\int_0^{\infty} x^{k-1}\, exp[\lambda\alpha(1-exp(x/\alpha)^{\beta})]\, dx. \qquad (3.1)$$

He expressed the k^{th} moment as simple derivatives of the incomplete gamma function.

$$E(X^k) = n\alpha^k exp(\lambda\alpha)\frac{\partial^{n-1}}{\partial\gamma^{n-1}}[(\lambda\alpha)^{-\gamma}\Gamma(\gamma,\lambda\alpha)]. \qquad (3.2)$$

Here, $n = k/\beta$, for $k = 1,2,3,\ldots$ where the derivative is evaluated as $\gamma \to 0$, and $\Gamma(a,x) = \int_x^{\infty} t^{a-1}\exp[-t]dt$ is the incomplete gamma function.

Equation (3.1) or equation (3.2) may compute the moments of the MWE(α,β,λ) distribution. For $n = 1,2,3,4$, equation (3.2) compute the moments as

$$E\left(X^{\beta}\right)=Ei(1,\lambda\alpha). \tag{3.3}$$

$$E\left(X^{2\beta}\right)=\frac{\pi^2}{12}+\frac{\gamma^2}{2}+\gamma\log\lambda+\gamma\log\alpha+\frac{(\log\lambda)^2}{2}$$

$$+\frac{(\log\alpha)^2}{2}+\log\lambda\log\alpha-\lambda\alpha_3F_3(1,1,1;2,2,2;-\lambda\alpha) \tag{3.4}$$

$$E\left(X^{3\beta}\right)=-\frac{2\eta(3)}{3}-\frac{\gamma^3}{3}-\frac{\pi^2\gamma}{6}-\left(\frac{\pi^2}{6}+\gamma^2\right)\log\lambda-\left(\frac{\pi^2}{6}+\gamma^2\right)\log\alpha$$

$$-2\gamma\log\lambda\log\alpha-\gamma(\log\lambda)^2-\gamma(\log\alpha)^2-\log\lambda(\log\alpha)^2-(\log\lambda)^2\log\alpha$$

$$-\frac{1}{3}(\log\lambda)^3-\frac{1}{3}(\log\alpha)^3+2\lambda\alpha_4F_4(1,1,1,1;2,2,2,2;-\lambda\alpha) \tag{3.5}$$

$$E\left(X^{4\beta}\right)=\frac{3\pi^4}{80}+2\zeta(3)\gamma+\frac{\pi^2\gamma^2}{4}+\frac{\gamma^4}{4}+\left(2\zeta(3)+\gamma^3+\frac{\pi^2\gamma}{2}\right)\log\lambda$$

$$+\left(2\zeta(3)+\gamma^3+\frac{\pi^2\gamma}{2}\right)\log\alpha+\left(3\gamma^2+\frac{\pi^2}{2}\right)\log\lambda\log\alpha$$

$$+\left(\frac{3\gamma^2}{2}+\frac{\pi^2}{4}\right)(\log\lambda)^2+\left(\frac{3\gamma^2}{2}+\frac{\pi^2}{4}\right)(\log\alpha)^2+3\gamma\log\lambda(\log\alpha)^2$$

$$+3\gamma(\log\lambda)^2\log\alpha+\gamma(\log\lambda)^3+\gamma(\log\alpha)^3+\log\lambda(\log\alpha)^3$$

$$+\frac{3}{2}(\log\lambda)^2(\log\alpha)^2+(\log\lambda)^3\log\alpha+\frac{1}{4}(\log\lambda)^4$$

$$+\frac{1}{4}(\log\alpha)^4-6\lambda\alpha_5F_5(1,1,1,1,1;2,2,2,2,2;-\lambda\alpha). \tag{3.6}$$

Here,

$Ei(n,x)=\int_1^{\infty}\frac{\exp(-xt)}{t^n}dt$ is the exponential integral function.

$F_q^p\left(a_1,\dots\dots,a_p;b_1,\dots,b_q;x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{x^k}{k!}$, is the generalized hyper geometric function.

$\gamma = 0.5772156649$ is the Euler's constant.

$\zeta(x) = \sum_{k=1}^{\infty} {}^{1}/_{k^x}$,

$\eta(x) = (1 - 2^{1-x})\zeta(x)$.

$(c)_k = c(c+1)\ldots(c+k-1)$ is the ascending factorial.

The formulas in equations (3.3)-(3.6) give the first four moments when $\beta = 1$, when $\beta = 2$ it gives the moments of order 2,4,6 and 8and when $\beta = 3$ the formulas give moments of order 3,6,9 and 12; and, so on.

3.4 Modified Weibull Type I Distribution

Sarhan and Zaindin (2009a) presented a new distribution called Modified Weibull Type I (MWD(I)(α, β, λ)) distribution which is a general form for some well-known distributions such as Exponential (E(α)), Rayleigh (R(λ)), linear failure rate (LFR(α, λ)) and Weibull(W(β, λ)) distributions and studied its different properties. This new distribution contains three parameters, two scale parameters α, λ and one shape parameter β and it has constant, increasing and decreasing hazard rate functions which are desirable for data analysis purposes. Sarhan and Zaindin (2009b) dealt with the problem of estimating the parameters of this distribution based on Type II censored data.

Zaindin (2010) estimated the unknown parameters of the MW (I) (α, β, λ) distribution based on grouped data and censored data. The point and asymptotic confidence of the unknown parameters are estimated by the maximum likelihood method.

Gasmi and Berzig (2011) developed the confidence estimation for the parameters of MW (I) (α, β, λ) distribution based on type I censored samples with, and without replacement.

The cumulative distribution function of the MW (I) (α, β, λ) distribution is

$$F(x) = 1 - exp\left(-\alpha x - \lambda x^{\beta}\right), x > 0, \lambda > 0, \alpha, \beta > 0, \alpha + \lambda > 0.$$

The probability density function is

$$f(x; \alpha, \beta, \lambda) = \left(\alpha + \lambda \beta x^{\beta - 1}\right) exp\left\{-\alpha x - \lambda x^{\beta}\right\}, x > 0, \alpha, \beta, \lambda > 0 \ .$$

The MW (I) (α, β, λ) distribution generalizes the LFR(α, λ) distribution at $\beta = 2$, the W(β, λ) distribution at $\alpha = 0$, the R(λ) distribution at $\alpha = 0, \beta = 2$ and the E(α) distribution at $\lambda = 0$.

The hazard function of the MW (I) (α, β, λ) distribution is

$$h(x) = \alpha + \lambda \beta x^{\beta - 1}.$$

The hazard function will be constant when $\beta = 1$ and when $\beta < 1$ it will be a decreasing function, while it will be an increasing function when $\beta > 1$.

The quantile x_q of the MW (I) (α, β, λ) distribution is a real solution of

$$\lambda x_q^{\beta} + \alpha x_q + \ln(1 - q) = 0.$$

This equation has no closed form solution in x_q. So, a numerical technique such as Newton-Raphson method will be used to get the quantile. When $q = 0.5$, the median can be obtained.

Sarhan and Zaindin (2009a) derived the quantile x_q for the special cases:

1. When $\beta = 2$ the MW (I) (α, β, λ) distribution reduces to linear failure rate LFR $(\alpha,\ \lambda)$ distribution with $x_q = \frac{1}{2\lambda}\left\{-\alpha + \sqrt{\alpha^2 - 4\lambda \ln(1-q)}\right\}$.
2. When $\alpha = 0$, the MW (I) (α, β, λ) distribution becomes Weibull W (β, λ) distribution with $x_q = \left\{-\frac{1}{\lambda} + \ln(1-q)\right\}^{1/\beta}$.
3. When $\alpha = 0$, $\beta = 2$, the MW(I)(α, β, λ) distribution becomes Rayleigh R(λ) distribution with $x_q = \sqrt{\left\{\frac{1}{\lambda} + \ln(1-q)\right\}}$.
4. When $\lambda = 0$, the MW(I)(α, β, λ) distribution reduces to exponential E(α) distribution with $x_q = -\frac{\ln(1-q)}{\alpha}$.
5. When $\beta = 1$, MW(I)(α, β, λ) distribution reduces exponential to E(□ + □) distribution with □□ $= -\frac{\ln(1-□)}{(□+□)}$.

The mode of the MW (I) (□, □, □) distribution is as a solution of the following nonlinear equation in□.

$$\alpha^2 + \lambda^2\beta^2 x^{2(\beta-1)} + 2\alpha\lambda\beta x^{\beta-1} - \lambda\beta(\beta-1)x^{\beta-2} = 0.$$

The k^{th} moment of X, say μ'_k is given by

$$\mu'_k = \begin{cases} \sum_{i=0}^{\infty} \frac{(-\lambda)^i}{i!}\left[\frac{\Gamma(i\beta+k+1)}{\alpha^{i\beta+k}} + \lambda\beta\frac{\Gamma(i\beta+k+\beta)}{\alpha^{i\beta+k+\beta}}\right] & for\ \alpha, \lambda > 0 \\ \frac{\Gamma\left(\frac{k}{\beta+1}\right)}{\lambda^{k/\beta}} & for\ \alpha = 0, \lambda > 0 \\ \frac{\Gamma(k+1)}{\alpha^k} & for\ \alpha > 0, \lambda = 0 \end{cases}$$

The measures of skewness $\alpha^*$and kurtosis λ^* of the MW (I) (α, β, λ) distribution are calculated for different values of β when $\alpha = 1$ and $\lambda = 0.5$. It is observed that α^* and λ^* first increase and then start decreasing. In addition, α^*takes negative values when β becomes large.

The moment generating function takes the form

$$M(t) = \begin{cases} \sum_{i=0}^{\infty} \frac{(-\lambda)^i}{i!} \left[\frac{\alpha \lceil (i\beta+1)}{(\alpha-t)^{i\beta+1}} + \lambda\beta \frac{\Gamma((i+1)\beta)}{(\alpha-t)^{(i+1)\beta}} \right] & for\ \alpha, \lambda > 0\ , \alpha > t \\ \frac{\sum_{i=0}^{\infty} t^i \lceil \left(\frac{i}{\beta+1}\right)}{\lambda^{i/\beta}} & for\ \alpha = 0, \lambda > 0 \\ \frac{\alpha}{\alpha-t} & for\ \alpha > 0, \lambda = 0, \alpha > t \end{cases}$$

The maximum likelihood estimates of the unknown parameters $\alpha, \beta\ and\ \lambda$ are derived based on complete sample. The likelihood function is

$$L = \prod_{i=1}^{n} \left(\alpha + \lambda\beta x_i^{\beta-1}\right) exp\left\{-\alpha x_i - \lambda x_i^{\beta}\right\}$$

The log likelihood function

$$\ln L = \sum_{i=1}^{n} \ln\left(\alpha + \lambda\beta x_i^{\beta-1}\right) - \alpha \sum_{i=1}^{n} x_i - \lambda \sum_{i=1}^{n} x_i^{\beta}\ .$$

Computing the first partial derivative of the log likelihood function with respect to $\alpha, \beta\ and\ \lambda$ and setting the results equal zeros, gives

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{\left(\alpha + \lambda\beta x_i^{\beta-1}\right)} - \sum_{i=1}^{n} x_i = 0,$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n} \frac{\beta x_i^{\beta-1}}{\left(\alpha + \lambda\beta x_i^{\beta-1}\right)} - \sum_{i=1}^{n} x_i^{\beta} = 0,$$

$$\frac{\partial \ln L}{\partial \beta} = \sum_{i=1}^{n} \frac{x_i^{\beta-1}(1+\beta \ln(x_i))}{\left(\alpha + \lambda\beta x_i^{\beta-1}\right)} - \sum_{i=1}^{n} x_i^{\beta} \ln(x_i) = 0.$$

The solution of this system is not possible in a closed form. So, the solution is obtained numerically.

The approximate confidence intervals of the parameters based on the asymptotic distributions of their maximum likelihood estimators are derived. The second partial derivatives of the log likelihood function for the observed information matrix of $\alpha, \beta\ and\ \lambda$ are

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -I_{11} = \sum_{i=1}^{n} \frac{1}{\left(\alpha + \lambda\beta x_i^{(\beta-1)}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} = -I_{12} = \sum_{i=1}^{n} \frac{\lambda x_i^{(\beta-1)}(1+\beta\ln(x_i))}{\left(\alpha+\lambda\beta x_i^{(\beta-1)}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} = -I_{13} = \sum_{i=1}^{n} \frac{\beta x_i^{(\beta-1)}}{\left(\alpha+\lambda\beta i^{(\beta-1)}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial\beta^2} = -I_{22} = \lambda\sum_{i=1}^{n} \frac{x_i^{(\beta-1)}\left(\alpha\beta\ln^2(x_i)+2\alpha\ln(x_i)-\lambda x_i^{\beta-1}\right)}{\left(\alpha+\lambda\beta i^{(\beta-1)}\right)^2} + \lambda\sum_{i=1}^{n} x_i^{\beta}\ln^2(x_i),$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\lambda} = -I_{23} = \sum_{i=1}^{n} \frac{\alpha x_i^{(\beta-1)}(1+\beta\ln(x_i))}{\left(\alpha+\lambda\beta x_i^{(\beta-1)}\right)^2} + \sum_{i=1}^{n} x_i^{\beta}\ln(x_i),$$

$$\frac{\partial^2 \ln L}{\partial\lambda^2} = -I_{33} = \sum_{i=1}^{n} \frac{\beta^2 x_i^{2(\beta-1)}}{\left(\alpha+\lambda\beta i^{(\beta-1)}\right)^2}.$$

The observed information matrix is given by

$$A = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

So that the Variance-Covariance matrix may be approximated as

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}^{-1}.$$

The asymptotic distribution of the maximum likelihood estimators is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}\right]. \quad (3.7)$$

The matrix V involves the parameters $\alpha, \beta\ and\ \lambda$. Replacing the parameters by their corresponding maximum likelihood estimators to estimate V, gives

$$\hat{V} = \begin{pmatrix} \hat{I}_{11} & \hat{I}_{12} & \hat{I}_{13} \\ \hat{I}_{21} & \hat{I}_{22} & \hat{I}_{23} \\ \hat{I}_{31} & \hat{I}_{32} & \hat{I}_{33} \end{pmatrix}^{-1} .$$

Here, $\hat{I}_{ij} = I_{ij}$ when $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$replaces$(\alpha, \beta, \lambda)$. By using equation (3.7), the approximate $100(1-\gamma)\%$ confidence intervals for α, β, λ are determined, respectively, as $\hat{\alpha} \pm z_{\frac{\theta}{2}}\sqrt{\hat{V}_{11}}$, $\hat{\beta} \pm z_{\frac{\theta}{2}}\sqrt{\hat{V}_{22}}$, and $\hat{\lambda} \pm z_{\frac{\theta}{2}}\sqrt{\hat{V}_{33}}$.

Here, z_θ is the upper θ^{th} percentile of the standard normal distribution.

Based on Type II censored data Sarhan and Zaindin (2009b) dealt with the problem of estimating the parameters of the MW (I) (α, β, λ) distribution. The maximum likelihood estimators were used to derive the point and interval estimates of the parameters.

The likelihood function of $\underline{x}$ is

$$L(\underline{x}) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left(\alpha + \lambda\beta x_i^{\beta-1}\right) \exp\{-\alpha T_1(1) - \lambda T_1(\beta)\},$$

Here,

$$T_1(\beta) = (n-r)x_r^{\beta} + \sum_i^r x_i^{\beta} .$$

The log-likelihood function takes the form

$$\ln L(\underline{x}) = C + \sum_{i=1}^{r} \ln\left(\alpha + \lambda\beta x_i^{\beta-1}\right) - \alpha T_1(1) - \lambda T_1(\beta).$$

The value $C = \ln n! - \ln(n-r)!$ is a constant.

Calculating the first partial derivatives of the log likelihood with respect to α, β, λ and equating each equation to zero gives the following system of nonlinear equations:

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{r} \frac{1}{\alpha + \lambda\beta x_i^{\beta-1}} - T_1(1) = 0,$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{r} \frac{\beta x_i^{\beta-1}}{\alpha+\lambda\beta x_i^{\beta-1}} - T_1(\beta) = 0,$$

$$\frac{\partial \ln L}{\partial \beta} = \lambda \sum_{i=1}^{r} \frac{x_i^{\beta-1}(1+\beta \ln(x_i))}{\alpha+\lambda\beta x_i^{\beta-1}} - \lambda T_1'(\beta) = 0,$$

Here,

$$T_1'(\beta) = (n-r)x_r^{\beta} \ln(x_r) + \sum_i^r x_i^{\beta} \ln(x_i).$$

These equations are solved numerically.

The approximate confidence intervals of the parameters based on the asymptotic distributions of the maximum likelihood estimators of the parameters α, β, λ based on type II censored data are derived. For the observed information matrix of α, β, λ they found the following second partial derivatives of the log likelihood function

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -A_{11} = \sum_{i=1}^{r} \frac{1}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = -A_{12} = \lambda \sum_{i=1}^{r} \frac{(1+\beta \ln(x_i))x_i^{\beta-1}}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = -A_{13} = \sum_{i=1}^{r} \frac{\beta x_i^{\beta-1}}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2},$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -A_{22} = \alpha\lambda \sum_{i=1}^{r} \frac{x_i^{\beta-1}\left(\alpha\beta \ln^2(x_i) + 2\alpha \ln(x_i) - \lambda x_i^{\beta-1}\right)}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2} + \lambda T_1''(\beta),$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = -A_{23} = \alpha \sum_{i=1}^{r} \frac{(1+\beta \ln(x_i))x_i^{\beta-1}}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2} + T_1'(\beta),$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -A_{33} = \sum_{i=1}^{r} \frac{\beta^2 x_i^{2(\beta-1)}}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2}.$$

Here,

$$T_1''(\beta) = (n-r)x_i^{\beta} \ln^2(x_i) + \sum_i^r x_i^{\beta} \ln^2(x_i).$$

The observed information matrix is

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

The approximate variance-covariance matrix is

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = A^{-1}$$

It is known that the asymptotic distribution of the parameters $\alpha, \beta \text{ and } \lambda$ is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \end{pmatrix} \sim N \left[\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \right].$$

Since V involves the parameters $\alpha, \beta \text{ and } \lambda$, the parameters will be replaced by the corresponding maximum likelihood estimators in order to estimate V as

$$\hat{V} = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix}.$$

Here, $\hat{A}_{ij} = A_{ij}$ when $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ replaces(α, β, λ). The approximate $100(1-\theta)\%$ confidence intervals for α, β, λ are $\hat{\alpha} \pm z_{\frac{\theta}{2}} \sqrt{\hat{V}_{11}}$, $\hat{\beta} \pm z_{\frac{\theta}{2}} \sqrt{\hat{V}_{22}}$, and $\hat{\lambda} \pm z_{\frac{\theta}{2}} \sqrt{\hat{V}_{33}}$.

Here, z_θ is the upper θ^{th} percentile of the standard normal distribution.

Sarhan and Zaindin (2009b) also derived the least square estimators (LSEs) of the three parameters α, β, λ. Given the observed lifetimes $x_1, x_2, \dots, x_r$ in a type II censored sample from the MW (I) (α, β, λ) distribution. The least squares estimates of the parameters α, β, λ denoted $\hat{\alpha}_R, \hat{\beta}_R, \hat{\lambda}_R$ can be obtained by minimizing the quantity Q with respect to α, β, λ, where

$$Q = \sum_{i=1}^{r} \left\{ y_i - \alpha x_i - \lambda x_i^{\beta} \right\}^2 .$$

Here, $y_i = -\ln S_e(x_i)$ and $\hat{S}_e(x_i)$ is the empirical estimate of the survivor function

$$S(x) = exp\,(-\alpha x - \lambda x^{\beta})$$

at the observation $x_i, i = 1,2, \dots, m$, given by $S_e(x_i) = \frac{i - 0.5}{r}$.

Solving the following non-linear equations gives $\hat{\alpha}_R, \hat{\beta}_R, \hat{\lambda}_R$

$$\sum_{i=1}^{r} y_i x_i - \alpha \sum_{i=1}^{r} x_i^2 - \lambda \sum_{i=1}^{r} x_i^{\beta+1} = 0, \quad (3.8)$$

$$\sum_{i=1}^{r} y_i x_i^{\beta} - \alpha \sum_{i=1}^{r} x_i^{\beta+1} - \lambda \sum_{i=1}^{r} x_i^{2\beta} = 0, \quad (3.9)$$

$$\sum_{i=1}^{r} y_i x_i^{\beta} \ln(x_i) - \alpha \sum_{i=1}^{r} x_i^{\beta+1} \ln(x_i) - \lambda \sum_{i=1}^{r} x_i^{2\beta} \ln(x_i) = 0. \quad (3.10)$$

The two equations (3.8) and (3.9) give

$$\hat{\alpha}_R = \frac{\sum_{i=1}^{r} y_i x_i^{\beta} \sum_{i=1}^{r} x_i^2 - \sum_{i=1}^{r} x_i^{\beta+1} \sum_{i=1}^{r} y_i x_i}{\sum_{i=1}^{r} x_i^{2\beta} \sum_{i=1}^{r} x_i^2 - \left(\sum_{i=1}^{r} x_i^{\beta+1}\right)^2},$$

$$\hat{\lambda}_R = \frac{\sum_{i=1}^{r} y_i x_i \sum_{i=1}^{r} x_i^{2\beta} - \sum_{i=1}^{r} x_i^{\beta+1} \sum_{i=1}^{r} y_i x_i^{\beta}}{\sum_{i=1}^{r} x_i^{2\beta} \sum_{i=1}^{r} x_i^2 - \left(\sum_{i=1}^{r} x_i^{\beta+1}\right)^2}.$$

Substituting $\hat{\alpha}_R, \hat{\lambda}_R$ into equation (3.10) and solving it numerically gives $\hat{\beta}_R$.

Zaindin (2010) derived the mean time-to-failure of the MW (I) (α, β, λ) distribution in the form:

$$MTTF = \frac{1}{\beta \lambda^{\frac{1}{\beta}}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\alpha}{\lambda^{\frac{1}{\beta}}} \right)^i \Gamma\left(\frac{i+1}{\beta}\right).$$

He derived the estimate of the unknown parameters of the MW (I) (α, β, λ) distribution based on grouped and censored data. First, letting $t = t_1, t_2, \dots, t_k$, where

$t_1 < t_2 < \cdots < t_k$ denote the predetermined inspection times with t_k representing the completion time of the test. Second, letting $t_0 = 0$ and $t_{k+1} = \infty$. Third, for i=1,2,..,k denoting by n_i the number of failures recorded in the time interval (t_{i-1}, t_i) and by n_{k+1} the number of censored units that have not failed by the end of the test.

The maximum likelihood function is

$$L = C \prod_{i=1}^{k} [P\{t_{i-1} < T \le t_i\}]^{ni} [P\{T > t_k\}]^{n_{k+1}} .$$

Here,

$C = \frac{n!}{\prod_{i=1}^{k+1} n_i!}$ is a constant,

$$P\{t_{i-1} < T \le t_i\} = F(t_i) - F(t_{i-1}),$$

$$P\{T > t_k\} = 1 - F(t_k).$$

Therefore, the likelihood function is

$$L = C \left[exp\left\{-\alpha t_k - \lambda t_k^{\beta}\right\}\right]^{n_{k+1}} \prod_{i=1}^{k} \left[exp\left\{-\alpha t_{i-1} - \lambda t_{i-1}^{\beta}\right\} - exp\left\{-\alpha t_i - \lambda t_i^{\beta}\right\}\right]^{n_i}.$$

The log likelihood function is

$$\ln L = \ln C - n_{k+1} \left[\alpha t_k + \lambda t_k^{\beta}\right]^{n_{k+1}} + \sum n_i \ln \left[exp\left\{-\alpha t_{i-1} - \lambda t_{i-1}^{\beta}\right\} - exp\left\{-\alpha t_i - \lambda t_i^{\beta}\right\}\right]$$

Let $A_i = \begin{cases} 0 & if \;\; i = 0 \\ e^{-\alpha t_i - \lambda t_i^{\beta}} & if \; i = 1, \dots k \\ 1 & if \; i = k+1 \end{cases}$

The first partial derivatives of the log likelihood function with respect to α, β, λ are

$$\frac{\partial \ln L}{\partial \alpha} = -n_{k+1} t_k - \sum_{i=1}^{k} n_i \frac{A_{i-1} t_{i-1} - A_i t_i}{A_{i-1} - A_i},$$

$$\frac{\partial \ln L}{\partial \beta} = -n_{k+1} \lambda t_k^{\beta} \ln(t_k) - \sum_{i=1}^{k} n_i \frac{A_{i-1} t_{i-1}^{\beta} \lambda \ln(t_{i-1}) - A_i t_i^{\beta} \lambda \ln(t_{i-1})}{A_{i-1} - A_i} ,$$

$$\frac{\partial \ln L}{\partial \lambda} = -n_{k+1} t_k^{\beta} - \sum_{i=1}^{k} n_i \frac{A_{i-1} t_{i-1}^{\beta} - A_i t_i^{\beta}}{A_{i-1} - A_i} .$$

The solution of the equations $\frac{\partial \ln L}{\partial \alpha} = 0, \frac{\partial \ln L}{\partial \beta} = 0, \frac{\partial \ln L}{\partial \lambda} = 0$ is not possible in a closed form. So, the maximum likelihood estimators are obtained numerically.

The approximate confidence intervals of the parameters based on the asymptotic distributions of the maximum likelihood estimators of the parameters are constructed.

The following are the second partial derivatives of the log likelihood function

$$I_{11} = \frac{\partial^2 \ln L}{\partial \alpha^2} = -\sum_{i=1}^{k} n_i \frac{A_i A_{i-1}(t_i - t_{i-1})^2}{(A_i - A_{i-1})^2},$$

$$I_{12} = \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = -\lambda \sum_{i=1}^{k} n_i \frac{A_i A_{i-1}\left(t_i^{\beta+1} \ln(t_i) - t_i^{\beta} t_{i-1} \ln(t_i) - t_i t_{i-1}^{\beta} \ln(t_{i-1}) + t_{i-1}^{\beta+1} \ln(t_{i-1})\right)}{(A_i - A_{i-1})^2},$$

$$I_{13} = \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = -\sum_{i=1}^{k} n_i \frac{A_i A_{i-1}\left(t_i^{\beta+1} - t_i^{\beta} t_{i-1} - t_i t_{i-1}^{\beta} + t_{i-1}^{\beta+1}\right)}{(A_i - A_{i-1})^2},$$

$$I_{22} = \frac{\partial^2 \ln L}{\partial \beta^2} = \lambda \sum_{i=1}^{k} n_i \{A_i A_{i-1} t_{i-1}^{\beta} \ln^2(t_{i-1})\left(1 - \lambda t_{i-1}^{\beta}\right)$$

$$+ A_i A_{i-1} t_i^{\beta} \ln^2(t_i)\left(1 - \lambda t_i^{\beta}\right) + 2\lambda A_i A_{i-1} t_i^{\beta}\, t_{i-1}^{\beta} \ln(t_i) \ln(t_{i-1})$$

$$+ A_i^2 t_i^{\beta} \ln(t_i) - A_{i-1}^2 t_{i-1}^{\beta} \ln^2(t_{i-1})\}/(A_i - A_{i-1})^2,$$

$$I_{23} = \frac{\partial^2 \ln L}{\partial \lambda \partial \beta} = -\sum_{i=1}^{k} n_i \{A_i A_{i-1} \lambda t_i^{\beta} t_i^{\beta+1}(\ln(t_{i-1}) + \ln(t_i))$$

$$+ A_i A_{i-1} t_{i-1}^{\beta} \ln(t_{i-1}) \left(1 - \lambda t_{i-1}^{\beta}\right)$$

$$+ A_i A_{i-1} t_i^{\beta} \ln(t_i) \left(1 - \lambda t_i^{\beta}\right) + A_i^2 t_i^{\beta} \ln(t_i)$$

$$- A_{i-1}^2 t_{i-1}^{\beta} \ln(t_{i-1})\}/(A_i - A_{i-1})^2,$$

$I_{33} = \frac{\partial^2 \ln L}{\partial \lambda^2} = -\sum_{i=1}^{k} n_i \frac{A_i A_{i-1}\left(t_i^{\beta} - t_{i-1}^{\beta}\right)^2}{(A_i - A_{i-1})^2}$.

The approximate $100(1-v)\%$ two sided confidence intervals for α, β, λ are, respectively$\hat{\alpha} \pm z_{\frac{v}{2}}\sqrt{I_{11}^{-1}}$, $\hat{\beta} \pm z_{\frac{v}{2}}\sqrt{I_{11}^{-1}}$, and $\hat{\lambda} \pm z_{\frac{v}{2}}\sqrt{I_{11}^{-1}}$, where, $z_{\frac{v}{2}}$ is the upper $\left(\frac{v}{2}\right)^{th}$ percentile of the standard normal distribution.

Gasmi and Berzig (2011) developed the estimation of the MW (I) (α, β, λ) distribution based on Type I censored samples without and with replacements. In the case of type I censoring without replacement N times are independently observed and the observation of the i^{th} item (I = 1,…, N) is censored at time T_i.

The likelihood function based on type I censored sample is

$$L = \left\{\prod_{i=1}^{n}\left(\alpha + \lambda\beta x_i^{\beta-1}\right) exp\left(-\alpha x_i - \lambda x_i^{\beta}\right)\right\}\left[exp\left(-\alpha T - \lambda T^{\beta}\right)\right]^{N-n} .$$

The log likelihood function is

$$\ln L = \sum_{i=1}^{n}\left[\ln\left(\alpha + \lambda\beta x_i^{\beta-1}\right) - \alpha x_i - \lambda x_i^{\beta}\right] + (N-n)\left(-\alpha T - \lambda T^{\beta}\right) .$$

Calculating the first partial derivatives of the log likelihood function with respect to α, β, λ and equating each equation to zero give the following system of nonlinear equations:

$$\sum_{i=1}^{n} \frac{1}{\alpha + \lambda\beta x_i^{\beta-1}} - \sum_{i=1}^{n} x_i - (N-n)T = 0, \qquad (3.11)$$

$$\sum_{i=1}^{n} \frac{\beta x_i^{\beta-1}}{\alpha + \lambda\beta x_i^{\beta-1}} - \sum_{i=1}^{n} x_i^{\beta} - (N-n)T^{\beta} = 0, \qquad (3.12)$$

$$\lambda \sum_{i=1}^{n} \frac{x_i^{\beta-1}(1+\beta \ln x_i)}{\alpha + \lambda\beta x_i^{\beta-1}} - \lambda \sum_{i=1}^{n} x_i^{\beta} \ln x_i - \lambda(N-n)T^{\beta} \ln T = 0. \qquad (3.13)$$

The equations (3.11) - (3.13) are solved numerically in α, β, λ.

Gasmi and Berzig (2011) obtained the estimation of the Fisher information matrix and asymptotic confidence bounds. They found the second partial derivatives of the log likelihood function as:

$$-\frac{\partial^2 \ln L}{\partial \alpha^2} = \sum_{i=1}^{n} \frac{1}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2},$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} \frac{\beta x_i^{\beta-1}}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2},$$

$$-\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \frac{\lambda x_i^{\beta-1}(1+\beta \ln x_i)}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2},$$

$$-\frac{\partial^2 \ln L}{\partial \lambda^2} = \sum_{i=1}^{n} \frac{\beta^2 x_i^{2(\beta-1)}}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2},$$

$$-\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = -\sum_{i=1}^{n} \frac{\alpha x_i^{\beta-1}(1+\beta \ln x_i)}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2} + \sum_{i=1}^{n} x_i^{\beta} \ln x_i + (N-n)T^{\beta} \ln T,$$

$$-\frac{\partial^2 \ln L}{\partial \beta^2} = -\lambda \sum_{i=1}^{n} \frac{x_i^{\beta-1}(\alpha\beta \ln^2 x_i + 2\alpha \ln x_i - \lambda x_i^{\beta-1})}{\left(\alpha+\lambda\beta x_i^{\beta-1}\right)^2} + \lambda \sum_{i=1}^{n} x_i^{\beta} \ln^2 x_i$$

$$+\lambda(N-n)T^{\beta} \ln^2 T$$

The observed information matrix A is

$$A = \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \beta^2} & -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 \ln L}{\partial \gamma \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} & -\frac{\partial^2 \ln L}{\partial \lambda^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

The variance-covariance matrix is

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = A^{-1}.$$

The asymptotic confidence intervals of the parameters α, β and λ are $\hat{\alpha} \pm z_{\frac{v}{2}}\sqrt{\hat{V}_{11}}$, $\hat{\beta} \pm z_{\frac{v}{2}}\sqrt{\hat{V}_{22}}$, and $\hat{\lambda} \pm z_{\frac{v}{2}}\sqrt{\hat{V}_{33}}$. Here, $z_{\frac{v}{2}}$ is the upper $\left(\frac{v}{2}\right)^{th}$ percentile of the standard normal distribution.

Gasmi and Berzig (2011) improved the confidence regions for small samples based on the likelihood ratio. The log-likelihood ratio

$q = 2\left\{\ln L\left(x; \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right) - \ln L(x; \alpha, \beta, \lambda)\right\}$ converges in-distribution to a central χ^2-distribution with 3 degrees of freedom. They developed the confidence estimation for the parameters of the MW (I) (α, β, λ) distribution based on type I censored samples without replacement. They observed N independent items, after each failure the item is immediately replaced by a new one and the observation continued up to the time T_i, i=1,..,N. The likelihood function for the renewal process is

$$L = \prod_{i=1}^{N}\left\{\prod_{j=1}^{d_i}\left(\alpha + \lambda\beta x_{ij}^{\beta}\right)\exp\left(-\alpha x_{ij} - \lambda x_{ij}^{\beta}\right)\right\}\exp\left(-\alpha R_i - \lambda R_i^{\beta}\right).$$

Here, $R_i = T_i - \sum_{j=1}^{d_i} x_{ij}$ is the rest-time of the observation, d_i is the number of failures of the i^{th} realization of the process and $x_i = (x_{i1}, x_{i2}, \dots, x_{id_i})$ denotes the distance between failures. The log-likelihood function is

$$\ln L = \sum_{i=1}^{N}\sum_{j=1}^{d_i}\ln(\alpha + \lambda\beta x_{ij}^{\beta}) - \sum_{i=1}^{N}\sum_{j=1}^{d_i}\ln(\alpha x_{ij} + \lambda\beta x_{ij}^{\beta}) - \sum_{i=1}^{N}(\alpha R_i - \lambda R_i^{\beta}).$$

The maximum likelihood estimators are obtained by using a suitable numerical method to solve the following system of non linear equations:

$$\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{1}{\alpha+\lambda\beta x_{ij}^{\beta-1}}-\sum_{i=1}^{N}\sum_{j=1}^{d_i}x_{ij}-\sum_{i=1}^{N}R_i=0,$$

$$\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{\beta x_{ij}^{\beta-1}}{\alpha+\lambda\beta x_{ij}^{\beta-1}}-\sum_{i=1}^{N}\sum_{j=1}^{d_i}x_{ij}^{\beta}-\sum_{i=1}^{N}R_i^{\beta}=0,$$

$$\lambda\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{x_{ij}^{\beta-1}(1+\beta\ln x_{ij})}{\alpha+\lambda\beta x_{ij}^{\beta-1}}-\lambda\sum_{i=1}^{N}\sum_{j=1}^{d_i}x_{ij}^{\beta}\ln x_{ij}-\lambda\sum_{i=1}^{N}R_i^{\beta}\ln R_i=0.$$

In this case, Gasmi and Berzig (2011) derived the observed Fisher information matrix for the parameters with the second partial derivatives of the log-likelihood function as:

$$-\frac{\partial^2\ln L}{\partial\alpha^2}=\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{1}{\left(\alpha+\lambda\beta x_{ij}^{\beta-1}\right)^2},$$

$$-\frac{\partial^2\ln L}{\partial\alpha\partial\beta}=\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{\beta x_{ij}^{\beta-1}(1+\beta\ln x_{ij})}{\left(\alpha+\lambda\beta x_{ij}^{\beta-1}\right)^2},$$

$$-\frac{\partial^2\ln L}{\partial\alpha\partial\lambda}=\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{\beta x_{ij}^{\beta-1}}{\left(\alpha+\lambda\beta x_{ij}^{\beta-1}\right)^2},$$

$$-\frac{\partial^2\ln L}{\partial\beta^2}=-\lambda\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{x_{ij}^{\beta-1}(\alpha\beta\ln^2 x_{ij}+2\alpha\ln x_{ij}-\lambda x_{ij}^{\beta-1})}{\left(\alpha+\lambda\beta x_{ij}^{\beta-1}\right)^2}+\lambda\sum_{i=1}^{N}\sum_{j=1}^{d_i}x_{ij}^{\beta}\ln^2 x_{ij}+$$

$$\lambda\sum_{i=1}^{N}R_i^{\beta}\ln^2 R_i,$$

$$-\frac{\partial^2\ln L}{\partial\lambda\partial\beta}=\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{\alpha x_{ij}^{\beta-1}(1+\beta\ln x_{ij})}{\left(\alpha+\lambda\beta x_{ij}^{\beta-1}\right)^2}+\sum_{i=1}^{N}\sum_{j=1}^{d_i}x_{ij}^{\beta}\ln x_{ij}+\sum_{i=1}^{N}R_i^{\beta}\ln R_i,$$

$$-\frac{\partial^2\ln L}{\partial\lambda^2}=\sum_{i=1}^{N}\sum_{j=1}^{d_i}\frac{\beta^2 x_{ij}^{2(\beta-1)}}{\left(\alpha+\lambda\beta x_{ij}^{\beta-1}\right)^2}.$$

The observed information matrix A is

$$A = \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \beta^2} & -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} & -\frac{\partial^2 \ln L}{\partial \lambda^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

The approximate variance-covariance matrix is

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = A^{-1}.$$

The asymptotic confidence intervals of the parameters α, β and λ are given by $\hat{\alpha} \pm z_{\frac{v}{2}}\sqrt{\hat{V}_{11}}$, $\hat{\beta} \pm z_{\frac{v}{2}}\sqrt{\hat{V}_{22}}$, and $\hat{\lambda} \pm z_{\frac{v}{2}}\sqrt{\hat{V}_{33}}$. Here, z_v is the upper v^{th} percentile of the standard normal distribution.

In this case, the confidence regions for small samples were constructed based on the likelihood ratio and with the log-likelihood ratio $q = 2\{\ln L\left(x; \hat{\alpha}, \hat{\beta}, \hat{\lambda}\right) - \ln L(x; \alpha, \beta, \lambda)\}$ that converges in distribution to a central χ^2- distribution with 3 degrees of freedom.

3.5 Modified Weibull Type II Distribution

The modified Weibull Type II (MW (II) (α, β, λ)) distribution has been recently introduced by Lai *et al*. (2003) as an extension of the Weibull model. The model can be considered as a useful three-parameter generalization of the Weibull distribution. The bathtub-shaped hazard rate function was proposed and they derived the model as a limiting case of the Beta integrated model and have both the Weibull distribution and Type I extreme value distribution as special cases.

Lai *et al*. (2003) estimated the parameters based on Weibull probability paper (WPP) plot and they studied the model characterization based on WPP plot.

Ng (2005) studied the estimation of parameters of the MW (II) (α, β, λ) distribution based on a progressively type II censored sample and derived the likelihood equations and the maximum likelihood estimators. The model's parameters based on least squares fit of a multiple linear regression on WPP plot (LSRE) are compared with the maximum likelihood estimators via Monet Carlo simulation. The observed Fisher information matrix as, well as the asymptotic variance-covariance matrix of the maximum likelihood estimators, were derived. He constructed approximate confidence intervals for the parameters based on standard normal approximation to the asymptotic distribution of the maximum likelihood estimation and the logarithmic transformed maximum likelihood estimation.

In (2008) Perdona *et al*. investigated the properties of the MW (II) (α, β, λ) distribution, a three-parameter distribution which allows U-shaped hazard to be accommodated. They presented the inference of the parameters based on both complete and censored samples. Different parameterizations as well as interval estimation for the parameters of this model were discussed.

Alwasel (2009) studied the competing risk model in the presence of incomplete and censored data when the causes of failures obey the MW (II) (α, β, λ) distribution. The maximum likelihood estimators of different parameters were derived. Also, asymptotic two-sided confidence intervals were obtained.

The cumulative distribution function of the MW (II) (α, β, λ) distribution is

$$F(x) = 1 - exp\left[-\lambda x^{\beta} e^{\alpha x}\right], x > 0, \alpha > 0, \beta \geq 0, \lambda > 0 .$$

The probability density function is

$$f(x; \alpha, \beta, \lambda) = \lambda(\beta + \alpha x) x^{\beta - 1} \exp(\alpha x) \exp\left[-\lambda x^{\beta} e^{\alpha x}\right].$$

The hazard function is

$h(x) = \lambda(\beta + \alpha x)x^{\beta-1}\exp(\alpha x)$.

The shape of the hazard function $h(x)$ depends only on β in $x^{\beta-1}$ because the remaining two parameters have no influence.

When $\beta \geq 1$, $h(x)$ is increasing in x, $h(0) = 0$ if $\beta > 1$; $h(0) = \lambda\beta$ if $\beta = 1$ and $h(x) \to \infty$ as $x \to \infty$.

When $0 < \beta < 1$, the hazard function initially decreases and then increases with x, implying a bathtub shape. For it $h(x) \to \infty\ as\ x \to 0$, and $h(x) \to \infty\ as\ x \to \infty$.

The derivative of the hazard function $h(x)$ intersects the x-axis only once, at x^* for $x > 0$. The hazard function $h(x)$ is decreasing for $x < x^*$, and increasing for $x > x^*$, where $x^* = (\sqrt{\beta} - \beta)/\alpha$.

The MW (II) (α, β, λ) distribution is related to the two-parameter W (β, λ) distribution for $\alpha = 0$. When $\beta = 2$ and $= 0$, it reduces to the R (λ) distribution. When $\beta = 0$, the model reduces to the extreme-value Type I EXT (I) (α, λ) distribution.

Lai *et al*. (2003) discussed the problem of determining whether a given data set can be adequately modeled by MW (II) (α, β, λ) distribution by WPP plot.

As for any traditional lifetime distribution the model's parameters must be estimated based on actual data. Lai *et al*. (2003) estimated the parameters based on WPP plot, the method of percentile, and the maximum likelihood method.

The likelihood function is easy to be derived. For complete data the log likelihood function is

$$\ln L = n\ln\alpha + \sum_{i=1}^{n}\ln(\beta+\alpha x_i) + (\beta-1)\ln(\sum_{i=1}^{n} x_i) + \alpha\sum_{i=1}^{n} x_i - \lambda\sum_{i=1}^{n}\left(x_i^{\beta}e^{\alpha x_i}\right).$$

Calculating the first partial derivatives of the log likelihood function with respect to α, β, λ and equating each derivative to zero gives the following equations:

$$\frac{\partial \ln L}{\partial\alpha} = \sum_{i=1}^{n}\frac{1}{\beta+\alpha x_i} + \sum_{i=1}^{n} x_i - \lambda\left(\sum_{i=1}^{n} x_i^{\beta+1}e^{\alpha x_i}\right) = 0,$$

$$\frac{\partial \ln L}{\partial\beta} = \sum_{i=1}^{n}\frac{1}{\beta+\alpha x_i} + \ln(\sum_{i=1}^{n} x_i) - \lambda\left(\sum_{i=1}^{n} x_i^{\beta}e^{\alpha x_i}\ln x_i\right) = 0,$$

$$\frac{\partial \ln L}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{\beta}e^{\alpha x_i} = 0.$$

From the third equation it follows that

$$\hat{\lambda} = n\left(\sum_{i=1}^{n}\left(x_i^{\beta}e^{\alpha x_i}\right)\right)^{-1}.$$

The remaining two equations need to be solved numerically to get $\hat{\alpha}, \hat{\beta}$.

Ng (2005) estimated the parameters of the MW (II) (α, β, λ) distribution by the WPP plot and the maximum likelihood method based on a progressively type II censored sample. He first discussed the problem of the point estimation of the models parameters based on least square regression on WPP plot.

The likelihood function based on a progressively type II censored sample is

$$L = C\prod_{i=1}^{m}\lambda(\beta+\alpha x_{i:m:n})x_{i:m:n}^{\beta-1}\exp\left[\alpha x_{i:m:n} - \lambda x_{i:m:n}^{\beta-1}e^{\alpha x_{i:m:n}}(R_i+1)\right].$$

Here, $x_{1:m:n} < \cdots < x_{m:m:n}$ are the observed values of such a progressively type II censored sample,

$C = n(n-1-R_1)(n-2-R_1-R_2)\ldots(n-m+1-R_1-\cdots-R_{m-1})$ and $(R_1, \ldots\ldots, R_m)$ is the progressively scheme.

The log likelihood function is given by

$$\ln L = k + m\ln\lambda + \sum_{i=1}^{m}\ln(\beta + \alpha x_{i:m:n}) + (\beta - 1)\sum_{i=1}^{m}\ln x_{i:m:n}$$

$$+\alpha\sum_{i=1}^{m} x_{i:m:n} - \lambda\sum_{i=1}^{m} x_{i:m:n}^{\beta}\, e^{\alpha x_{i:m:n}}(R_i + 1).$$

Here, K is a constant.

Calculating the first partial derivatives of the log likelihood function with respect to each of the parameters α, β, λ and equating derivative equations to zero gives the following system of nonlinear equations:

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{m}\frac{x_{i:m:n}}{\beta + \alpha x_{i:m:n}} + \sum_{i=1}^{m} x_{i:m:n} - \lambda\sum_{i=1}^{m} x_{i:m:n}^{\beta+1}\, e^{\alpha x_{i:m:n}}(R_i + 1) = 0,$$

$$\frac{\partial \ln L}{\partial \beta} = \sum_{i=1}^{m}\frac{1}{\beta + \alpha x_{i:m:n}} + \sum_{i=1}^{m}\ln x_{i:m:n} - \lambda\sum_{i=1}^{m} x_{i:m:n}^{\beta+1}\, e^{\alpha x_{i:m:n}}(R_i + 1) = 0,$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} x_{i:m:n}^{\beta} e^{\alpha x_{i:m:n}}(R_i + 1) = 0.$$

From the third equation, follows

$$\hat{\lambda} = \frac{m}{\sum_{i=1}^{m} x_{i:m:n}^{\beta} e^{\alpha x_{i:m:n}}(R_i+1)}.$$

The first and second equations will be solved numerically to get $\hat{\beta}, \hat{\alpha}$.

For the observed Fisher information matrix Ng (2005) derived the following second partial derivatives of the log likelihood function:

$$I_{11} = -\frac{\partial^2 \ln L}{\partial \alpha^2} = \sum_{i=1}^{m}\frac{x_{i:m:n}^2}{(\beta + \alpha x_{i:m:n})^2} + \lambda\sum_{i=1}^{m} x_{i:m:n}^{\beta+2}\, e^{\alpha x_{i:m:n}}(R_i + 1)\ ,$$

$$I_{12} = -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \sum_{i=1}^{m}\frac{x_{i:m:n}^2}{(\beta + \alpha x_{i:m:n})^2} + \lambda\sum_{i=1}^{m} x_{i:m:n}^{\beta}(\ln x_{i:m:n})\, e^{\alpha x_{i:m:n}}(R_i + 1)\ ,$$

$$I_{13} = -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{m} x_{i:m:n}^{\beta+1}\, e^{\alpha x_{i:m:n}}(R_i + 1),$$

$$I_{22} = -\frac{\partial^2 \ln L}{\partial \beta^2} = \sum_{i=1}^{m} \frac{1}{(\beta+\alpha x_{i:m:n})^2} + \alpha \sum_{i=1}^{m} x_{i:m:n}^{\beta} (\ln x_{i:m:n})^2 \, e^{\alpha x_{i:m:n}} (R_i + 1),$$

$$I_{23} = -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = \sum_{i=1}^{m} x_{i:m:n}^{\beta} (\ln x_{i:m:n}) \, e^{\alpha x_{i:m:n}} (R_i + 1) \; ,$$

$$I_{33} = -\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{m}{\lambda^2} \, .$$

The observed Fisher information matrix is

$$A = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

The matrix A can be inverted to obtain a local estimate of the asymptotic variance-covariance matrix of the maximum likelihood estimators

$$V = A^{-1} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}.$$

The asymptotic confidence intervals for the parameters α, β, λ are given respectively, by $\hat{\alpha} \pm z_{\frac{v}{2}} \sqrt{\hat{V}_{11}}$, $\hat{\beta} \pm z_{\frac{v}{2}} \sqrt{\hat{V}_{22}}$, and $\hat{\lambda} \pm z_{\frac{v}{2}} \sqrt{\hat{V}_{33}}$, where, z_v is the upper v^{th} percentile of the standard normal distribution.

In (2008) Perdona *et al*. derived the log likelihood function based on censored samples by considering a sample of independent random variables $X_1, X_2, \dots, X_n$ associated with survival times, and $c_1, c_2, \dots, c_n$ associated with censored times.

Let $T_i = \min\,(x_i, c_i)$ and let $\delta_i = I(x_i \le c_i)$ be the censoring indicator variable. The obtained log likelihood function can be written in the form:

$$\ln L = \sum_{i=1}^{n} \delta_i \,[\log(\lambda) + \log(\beta + \alpha t_i) + (\beta - 1)\log(t_i) + \alpha t_i] - \lambda \sum_{i=1}^{n} t_i^{\beta} e^{\alpha t_i} \, .$$

The first partial derivatives of the parameters α, β, λ are

$$\frac{\partial \ln L}{\partial \alpha} = \left[\sum_{i=1}^{n} \frac{\delta_i t_i}{\beta+\alpha t_i} + \sum_{i=1}^{n} \delta_i\, t_i\right] - \lambda \sum_{i=1}^{n} t_i^{\beta+1}\, e^{\alpha t_i},$$

$$\frac{\partial \ln L}{\partial \beta} = \left[\sum_{i=1}^{n} \frac{\delta_i}{\beta+\alpha t_i} + \sum_{i=1}^{n} \delta_i \log t_i\right] - \lambda \sum_{i=1}^{n} t_i^{\beta} e^{\alpha t_i} \ln t_i,$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n} \frac{\delta_i}{\lambda} - \sum_{i=1}^{n} t_i{}^{\beta}\, e^{\alpha t_i}.$$

By equating these equations to zero, the parameters can be obtained by solving the resulting equations numerically.

Inference regarding $v' = (\alpha, \beta, \lambda)$ can be based on the properties of the maximum likelihood estimation for large samples as $\hat{v} \sim N(v, A^{-1}(v))$.

Here, $A(v)$ is the Fisher information matrix, which is estimated by $A^{-1}(\hat{v})$ when v is replaced by the maximum likelihood estimator $\hat{v}$.

The observed Fisher information matrix for $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ is given by

$$\begin{bmatrix} \sum_{i=1}^{n} \frac{\delta_i t_i^2}{A_i{}^2} + \lambda u_i(t_i;0) e^{\alpha x_i} t_i^2 & \sum_{i=1}^{n} \frac{\delta_i t_i}{A_i{}^2} + \lambda u_i(t_i;1) e^{\alpha t_i} t_i & \sum_{i=1}^{n} u_i(t_i;0) e^{\alpha t_i} t_i \\ \sum_{i=1}^{n} \frac{\delta_i t_i}{A_i{}^2} + \lambda u_i(t_i;1) e^{\alpha t_i} t_i & \sum_{i=1}^{n} \frac{\delta_i}{A_i{}^2} + \lambda u_i(t_i;2) e^{\alpha t_i} & \sum_{i=1}^{n} \lambda u_i(t_i;1) e^{\alpha t_i} t_i \\ \sum_{i=1}^{n} u_i(t_i;0) e^{\alpha t_i} t_i & \sum_{i=1}^{n} \lambda u_i(t_i;1) e^{\alpha t_i} t_i & \sum_{i=1}^{n} \frac{\delta_i}{\lambda^2} \end{bmatrix}_{v=\hat{v}}$$

Here $u_b(x_i; b) = x_i^{\beta} \ln^{b} x_i$ for $b = 0,1,2, ..$ and $A_i = \beta + \alpha x_i$ with $i = 1, \dots, n$.

Alwasel (2009) derived the maximum likelihood function based on incomplete and censored data. He assumed that there are two causes of failures and assumptions:

1. The random vectors (X_{1i}, X_{2i}), $i = 1, \dots, n$ are n independent and identically distributed.
2. The random variables X_{1i} *and* X_{2i} are independent for all $i = 1, \dots, n$ and $X_i = \min(X_{1i}, X_{2i})$

3. The r .v. X_{ij} has MW(II) $(\alpha_i, \beta_i, \lambda_i)$ distribution, j=1,2, i=1,2,..,n.
4. In the first m observations, observe the failure times and also causes of failure. Whereas for the successive (n-m) observations, observe only the failure times and not the causes of failure, that is the cause of failure is unknown. In the successive (N-n) observations, the systems are still alive at the end the project periods. The observed data will be:

   $$(X_1, \delta_1), (X_2, \delta_2), \dots, (X_m, \delta_m), (X_{m+1}, *), \dots, (X_n, *), (X_{n+1^*}, *), \dots, (X_{N^*}, *).$$

   Here, (X, δ) means the system has failed at time x due to cause δ, and $(x *, *)$
   means the system has tested until time x without failing (censored data). This set is denoted by Ω which can be categorized as a union of three disjoint classes Ω_1 , Ω_2 and Ω_3, where Ω_1 represents the set of data when the cause of system failure is known, while Ω_2 denotes the set of observation when the cause of system's failure is unknown and Ω_3 denotes the set of censored observations.

   Further, the set Ω_1 can be divided into two disjoint subsets of observation: Ω_{11} and Ω_{12}, where Ω_{1j} represents the set of all observations when the failure of the system is due to the cause j, j=1,2. Also assume that $|\Omega_i| = r_i$, $|\Omega_{ij}| = r_{ij}$, $m = r_1 = r_{11} - r_{12}$, $|\Omega_2| = r_2 =$ n-m and $|\Omega_3| = r_3 =$N-n.
5. The lifetimes are from the same population as in the complete data. That is, the population remains unchanged irrespective of the cause of failure.
6. Also, m and n are predetermined.

The likelihood function for the observed data is

$(x_1;\delta_1),\ldots,(x_m;\delta_m),(x_{m+1};*),\ldots,(x_n;*),(x_{n+1^*};*),\ldots,(x_{N^*};*)$

The maximum likelihood function based on incomplete and censored data is

$$L = \exp\left\{-\sum_{j=1}^{2}\lambda_j\sum_{i=1}^{N}x_i^{\beta_j}\,e^{\alpha_j x_i}\right\}\prod_{j=1}^{2}\lambda_j^{r_j}\ .$$

$$\prod_{x_i\in\Omega_{1j}}\left\{\left(\beta_j+\alpha_j x_i\right)x_i^{\beta_j-1}e^{\alpha_j x_i}\right\}.$$

$$\prod_{x_i\in\Omega_2}\left\{(\beta_1+\alpha_1 x_i)x_i^{\beta_1-1}e^{\alpha_1 x_i}+(\beta_2+\alpha_2 x_i)x_i^{\beta_2-1}e^{\alpha_2 x_i}\right\}.$$

The log likelihood function is

$$\ln L = \sum_{j=1}^{2}\left\{r_j\ln\lambda_j-\lambda_j\sum_{i=1}^{N}x_i^{\beta_j}\,e^{\alpha_j x_i}\right\}$$

$$+\sum_{j=1}^{2}\sum_{x_i\in\Omega_{1j}}\left\{\ln\left(\beta_j+\alpha_j x_i\right)+\left(\beta_j-1\right)\ln x_i+\alpha_j x_i\right\}$$

$$+\sum_{x_i\in\Omega_2}\ln\left\{(\beta_1+\alpha_1 x_i)x_i^{\beta_1-1}e^{\alpha_1 x_i}+(\beta_2+\alpha_2 x_i)x_i^{\beta_2-1}e^{\alpha_2 x_i}\right\}.$$

Equating the first partial derivatives with respect to $\alpha_l,\beta_l,\lambda_l$ to zero gives

$$\sum_{x_i\in\Omega_2}\frac{\sum_{j=1}^{2}\delta_{il}\left[1+\left(\beta_j+\alpha_j x_i\right)\ln x_i\right]x_i^{\beta_j-1}e^{\alpha_j x_i}}{\sum_{j=1}^{2}(\beta_j+\alpha_j x_i)x_i^{\beta_j-1}e^{\alpha_j x_i}}$$

$$-\lambda_l\sum_{i=1}^{N}x_i^{\beta_l}\,x_i\,e^{\alpha_l x_i}+\sum_{x_i\in\Omega_{1j}}\left[\frac{x_i}{\beta_l+\alpha_l x_i}+x_i\right]=0\ ,$$

$$\sum_{x_i\in\Omega_2}\frac{\sum_{j=1}^{2}\delta_{il}\left[1+\left(\beta_j+\alpha_j x_i\right)\ln x_i\right]x_i^{\beta_j-1}e^{\alpha_j x_i}}{\sum_{j=1}^{2}(\beta_j+\alpha_j x_i)x_i^{\beta_j-1}e^{\alpha_j x_i}}$$

$$-\lambda_l\sum_{i=1}^{N}x_i^{\beta_l}\ln x_i e^{\alpha_l x_i}+\sum_{x_i\in\Omega_{1j}}\left[\frac{1}{\beta_l+\alpha_l x_i}+\ln x_i\right]=0\ ,$$

$$\frac{r_l}{\lambda_l}-\sum_{i=1}^{N}x_i^{\beta_l}\,e^{\alpha_l x_i}=0\ .$$

Here, $\delta_{jl} = \begin{cases} 1 & if\ j = l \\ 0 & o.w \end{cases}$

This system of nonlinear equations has no closed form solution, so numerical technique is required. To get the MLEs of the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1\ and\ \lambda_2$. Alwasel (2009) developed the relative risk rate due to two causes 1 and 2 in a closed form. The relative risk rates, π_1 due to cause 1 and π_2 due to cause 2 are given respectively, by

$$\pi_1 = P[X_{1i} < X_{2i}] = 1 - \lambda_1 \int_0^\infty (\beta_1 + \alpha_1 x) x^{\beta_1 - 1} e^{\alpha_1 x - \sum_{j=1}^{2} \alpha_j x^{\beta_j} e^{\alpha_j x}}\, dx.$$

$$\pi_2 = P[X_{2i} < X_{1i}] = 1 - \pi_1 = \lambda_1 \int_0^\infty (\beta_1 + \alpha_1 x) x^{\beta_1 - 1} e^{\alpha_1 x - \sum_{j=1}^{2} \alpha_j x^{\beta_j} e^{\alpha_j x}}\, dx\ .$$

The above integrals have no closed solution. So numerical integration technique is required to get $\pi_1\ and\ \pi_2$. The maximum likelihood estimation of the relative risk π_1 and π_2 can be obtained by replacing the unknown parameters λ_j, β_j and $\alpha_j, j = 1,2$ by their maximum likelihood estimators.

Some special cases can be reached from the above results as follows:

1. For the exponential distributions case, by setting $\beta_j = 1$ and $\alpha_j = 0, j = 1,2$

   $\pi_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}, \quad \pi_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$

2. For the Weibull distributions case with the same shape parameters β by setting $\beta_1 = \beta_2 = \beta$, and $\alpha_j = 0$

   $\pi_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}, \ \pi_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$

which, is the same as for the exponential case.

The asymptotic distribution of the maximum likelihood estimator

$\hat{v} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\lambda}_1, \hat{\lambda}_2)$ is $N_6(v, A^{-1}(v))$.

The elements of the matrix $A^{-1}(\hat{v}) = [I_{ij}(\hat{v})]$, $i, j = 1,2, \dots, 6$, where

$$A^{-1}(\hat{v}) = \left[-\frac{\partial^2 \ln L}{\partial v_i \partial v_j}\right]_{v=\hat{v}}$$

The asymptotic confidence intervals of v is $\hat{v} \pm z_{\frac{v}{2}}\sqrt{A^{-1}(\hat{v})}$, where z_v is the upper v^{th} percentile of the standard normal distribution.

Chapter IV

Discretizing Continuous Distributions

4.1 Introduction

An important aspect of lifetime analysis is to find a lifetime distribution that can adequately describe the ageing behavior of the device concerned. Most of the lifetimes are continuous in nature. Hence, many continuous life distributions do exist in literature. On the other hand, discrete failure data are arising in several common situations. For example, the life length of a copier would be the total number of copies it produces. Using the discretizing approach, the discrete form of the general class of continuous distributions can obtained. For the discretized class, the reliability measures and the characteristics will be derived.

The discrete modified Weibull extension (DMWE), the discrete modified Weibull Type I (DMW (I)), and discrete modified Weibull Type II (DMWD (II)) distributions will be introduced in this Chapter. The distributional properties of these distributions will be discussed.

4.2 Discretizing General Class of Continuous Distributions

We consider a general class of continuous distributions and generate a discrete lifetime distribution based on a continuous distribution.

Let X is a positive random variable having a cumulative distribution function

$$F(x) = 1 - e^{-\lambda g(x)}, \tag{4.1}$$

where $g(x) = g(x; \alpha, \beta), g(0) = 0$.

The corresponding survival function is

$$S(x) = P(X > x)$$

$$= e^{-\lambda g(x)} \,. \tag{4.2}$$

Using the first discretizing method, introduced in Section 3.2 of Chapter III, then for every positive integer x the pmf of the discretized class is

$$P(x) = P(X = x) = S(x) - S(x+1) = e^{-\lambda g(x)} - e^{-\lambda g(x+1)}.$$

Equivalently, for $\theta = e^{-\lambda}, \lambda > 0$, the pmf $P(x)$ is

$$P(x) = \theta^{g(x)} - \theta^{g(x+1)}. \tag{4.3}$$

4.2.1 Properties of the Discretized Class of Continuous Distributions

The discretized general class of continuous distribution has the following properties:

- The cumulative distribution function is

$$F_1(x) = 1 - \theta^{g(x+1)} \,. \tag{4.4}$$

Proof:

$$F_1(x) = \sum_{t=0}^{x} P(t)$$

$$= \sum_{t=0}^{x} [\theta^{g(t)} - \theta^{g(t+1)}] \text{ , since } g(0) = zero$$

$$= \left[\theta^{g(0)} + \theta^{g(1)} + \cdots + \theta^{g(x)}\right] - \left[\theta^{g(1)} + \theta^{g(2)} + \cdots + \theta^{g(x+1)}\right]$$

$$= 1 - \theta^{g(x+1)}.$$

- The survival function is

$$S_1(x) = \theta^{g(x+1)}. \quad (4.5)$$

Proof:

$$S_1(x) = P(X > x)$$

$$= \sum_{t=x+1}^{\infty} [\theta^{g(t)} - \theta^{g(t+1)}]$$

$$= \theta^{g(x+1)} .$$

- The failure rate is

$$r_1(x) = 1 - \theta^{g(x+1)-g(x)}. \quad (4.6)$$

Proof:

$$r_1(x) = \frac{\theta^{g(x)} - \theta^{g(x+1)}}{\sum_{t=x}^{\infty} [\theta^{g(t)} - \theta^{g(t+1)}]}$$

$$= \frac{\theta^{g(x)} - \theta^{g(x+1)}}{\theta^{g(x)}}$$

$$= 1 - \theta^{g(x+1)-g(x)}.$$

- The second of failure rate function, defined for every positive integer x is

$$SRF_1(x) = [g(x) - g(x+1)] \ln \theta. \quad (4.7)$$

Proof:

$$SRF_1(x) = \ln \left[\frac{S_1(x-1)}{S_1(x)} \right]$$

$$= \ln\left[\frac{\theta^{g(x)}}{\theta^{g(x+1)}}\right]$$

$$= \ln\left[\theta^{g(x)-g(x+1)}\right]$$

$$= [g(x) - g(x+1)] \ln \theta.$$

- The residual reliability function at time x, defined for all $i \in N^*$ is

$$R_1(i|x) = \theta^{g(x+i+1)-g(x+1)}. \quad (4.8)$$

Proof:

$$R_1(i|x) = \frac{S_1(x+i)}{S_1(x)}$$

$$= \frac{\theta^{g(x+i+1)}}{\theta^{g(x+1)}}$$

$$= \theta^{g(x+i+1)-g(x+1)}.$$

- The cumulative hazard function, defined for every positive integer x, is

$$H_1(x) = x - \sum_{t=1}^{x} \theta^{g(t+1)-g(t)}. \quad (4.9)$$

Proof:

$$H_1(x) = \sum_{t=1}^{x} r_1(t)$$

$$= \sum_{t=1}^{x} \frac{\theta^{g(t)} - \theta^{g(t+1)}}{\sum_{i=t}^{\infty}[\theta^{g(i)} - \theta^{g(i+1)}]}$$

$$= \sum_{t=1}^{x} \left(\frac{\theta^{g(t)} - \theta^{g(t+1)}}{\theta^{g(t)}}\right)$$

$$= \sum_{t=1}^{x}\left(1 - \theta^{g(t+1)-g(t)}\right)$$

$$= x - \sum_{t=1}^{x} \theta^{g(t+1)-g(t)} \quad .$$

- The discretized general class of continuous distributions has an increasing failure rate (IFR), since the equivalent conditions IFR1 and IFR2 are satisfied, where

$IFR1$: $\{r_1(x)\}_{x\geq 1}$ is an increasing sequence.

$IFR2$: For all $i \in N^*$, $\{R_1(i|x)\}_{x\geq 1}$ is a decreasing sequence in x.

Proof of IFR1: Let $g(x)$ be an increasing continuous function in x. For $0 < \theta < 1$, so $\{\theta^{g(x)}\}$ is a decreasing sequence for all $x \geq 1$. Hence, for $x_1 < x_2$ we have

$$=> \theta^{g(x_1)} > \theta^{g(x_2)}$$

$$=> x_1 + 1 < x_2 + 1$$

$$=> \theta^{g(x_1+1)} > \theta^{g(x_2+1)}$$

$$=> \frac{\theta^{g(x_1+1)}}{\theta^{g(x_1)}} > \frac{\theta^{g(x_2+1)}}{\theta^{g(x_2)}}$$

$$=> 1 - \frac{\theta^{g(x_1+1)}}{\theta^{g(x_1)}} < 1 - \frac{\theta^{g(x_2+1)}}{\theta^{g(x_2)}}$$

$$=> r_1(x_1) < r_1(x_2).$$

Proof of IFR2: Let $g(x)$ be an increasing continuous function on x. Since $0 < \theta < 1$, then for $x_1 < x_2$, we have

$$=> x_1 + 1 < x_2 + 1$$

$$=> \theta^{g(x_1+1)} > \theta^{g(x_2+1)}$$

$$=> x_1 + i + 1 < x_2 + i + 1$$

$$=> \theta^{g(x_1+i+1)} > \theta^{g(x_2+i+1)}$$

$$=> \frac{\theta^{g(x_1+i+1)}}{\theta^{g(x_1+1)}} > \frac{\theta^{g(x_2+i+1)}}{\theta^{g(x_2+1)}}$$

$$=> R_1(i|x_1) < R_1(i|x_2).$$

- The discrete general class of continuous distributions has an increasing failure rate in average (IFRA), since the following equivalent conditions IFRA1 and IFRA2 are satisfied, where

IFRA1: $\left\{\left(S_1(x)\right)^{1/x}\right\}_{x\geq 1}$ is a decreasing sequence.

IFRA2: $\left\{\frac{H_1(x)}{x}\right\}_{x\geq 1}$ is an increasing sequence.

Proof of IFRA1: Let $g(x)$ be an increasing continuous function of x. For $0 < \theta < 1$, $\left\{\theta^{g(x)}\right\}$ is a decreasing sequence for all $x \geq 1$. Hence, for $x_1 < x_2$ we have

$=> x_1 + 1 < x_2 + 1$

$=> \theta^{g(x_1+1)} > \theta^{g(x_2+1)}$

$=> \left\{\theta^{g(x_1+1)}\right\}^{1/x_1} > \left\{\theta^{g(x_2+1)}\right\}^{1/x_2}$

$=> \{S_1(x_1)\}^{1/x_1} > \{S_1(x_2)\}^{1/x_2}$.

Proof of IFRA2: Let $g(x)$ be an increasing continuous function of x. For $0 < \theta < 1$, $\left\{\theta^{g(x)}\right\}$ is a decreasing sequence for all $x \geq 1$. For $t_1 < t_2$ and $x_1 < x_2$ we have

$=> \theta^{g(t_1)} > \theta^{g(t_2)}$

$=> \theta^{g(t_1+1)} > \theta^{g(t_2+1)}$, since $t_1 + 1 < t_2 + 1$

$=> \frac{\theta^{g(t_1+1)}}{\theta^{g(t_1)}} > \frac{\theta^{g(t_2+1)}}{\theta^{g(t_2)}}$

$=> -\frac{\theta^{g(t_1+1)}}{\theta^{g(t_1)}} < -\frac{\theta^{g(t_2+1)}}{\theta^{g(t_2)}}$

$=> -\sum_{t_1=1}^{x_1} \frac{\theta^{g(t_1+1)}}{\theta^{g(t_1)}} < -\sum_{t_2=1}^{x_2} \frac{\theta^{g(t_2+1)}}{\theta^{g(t_2)}}$, since $-\frac{1}{x_1} < -\frac{1}{x_2}$

$=> -\frac{1}{x_1}\sum_{t_1=1}^{x_1} \frac{\theta^{g(t_1+1)}}{\theta^{g(t_1)}} < -\frac{1}{x_2}\sum_{t_2=1}^{x_2} \frac{\theta^{g(t_2+1)}}{\theta^{g(t_2)}}$

$=> 1 - \frac{1}{x_1}\sum_{t_1=1}^{x_1} \frac{\theta^{g(t_1+1)}}{\theta^{g(t_1)}} < 1 - \frac{1}{x_2}\sum_{t_2=1}^{x_2} \frac{\theta^{g(t_2+1)}}{\theta^{g(t_2)}}$

$=> \frac{H_1(x_1)}{x_1} < \frac{H_1(x_2)}{x_2}$.

Another proof:

From the definition of the cumulative hazard function, it can be seen that it is an increasing function but not a probability, where it measures the total amount of risk that has been accumulated up to time x (Mario *et. al*. (2008)) and for $\frac{H_1(x)}{x}$ it is also increasing.

- The r^{th} moment of the discretized general class is

$$E(X^r) = \sum_{x=1}^{\infty}(x^r - (x-1)^r)\theta^{g(x)} , g(0) = zero . \tag{4.10}$$

Proof:

$$E(X^r) = \sum_{x=0}^{\infty} x^r p(x)$$

$$= \sum_{x=0}^{\infty} x^r (\theta^{g(x)} - \theta^{g(x+1)})$$

$$= \sum_{x=0}^{\infty} x^r \theta^{g(x)} - \sum_{x=0}^{\infty} x^r \theta^{g(x+1)}$$

$$= \left[0^r \theta^{g(0)} + 1^r \theta^{g(1)} + 2^r \theta^{g(2)} + 3^r \theta^{g(3)} + \cdots\right]$$

$$-\left[0^r \theta^{g(1)} + 1^r \theta^{g(2)} + 2^r \theta^{g(3)} + 3^r \theta^{g(4)} + \cdots\right]$$

$$= \theta^{g(1)} + (2^r - 1)\theta^{g(2)} + (3^r - 2^r)\theta^{g(3)} + (4^r - 3^r)\theta^{g(4)} + \cdots$$

$$= \sum_{x=1}^{\infty}(x^r - (x-1)^r)\theta^{g(x)}.$$

- The moment generating function of the discretized general class is

$$M_X(t) = 1 + \sum_{x=1}^{\infty}\left(e^{xt} - e^{(x-1)t}\right)\theta^{g(x)}, g(0) = zero. \tag{4.11}$$

Proof:

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} (\theta^{g(x)} - \theta^{g(x+1)})$$

$$= \sum_{x=0}^{\infty} e^{tx}\, \theta^{g(x)} - \sum_{x=0}^{\infty} e^{tx}\, \theta^{g(x+1)}$$

$$= \left[e^0\theta^{g(0)} + e^t\theta^{g(1)} + e^{2t}\theta^{g(2)} + e^{3t}\theta^{g(3)} + \cdots\right]$$

$$-\left[e^0\theta^{g(1)} + e^t\theta^{g(2)} + e^{2t}\theta^{g(3)} + e^{3t}\theta^{g(4)} + \cdots\right]$$

$$= \theta^{g(0)} + (e^t - 1)\theta^{g(1)} + (e^{2t} - e^t)\theta^{g(2)} + \cdots$$

$$= 1 + \sum_{x=1}^{\infty}\left(e^{xt} - e^{(x-1)t}\right)\theta^{g(x)} .$$

- The probability generating function of the discretized general class is

$$G_{[X]}(t) = 1 + (t-1)\sum_{x=1}^{\infty} t^{x-1}\, \theta^{g(x)} \ , g(0) = zero. \tag{4.12}$$

Proof:

$$G_{[X]}(t) = E(t^X) = \sum_{x=0}^{\infty} t^x\, P(x)$$

$$= \sum_{x=0}^{\infty} t^x\, (\theta^{g(x)} - \theta^{g(x+1)})$$

$$= \sum_{x=0}^{\infty} t^x\, \theta^{g(x)} - \sum_{x=0}^{\infty} t^x\, \theta^{g(x+1)}$$

$$= \left[\theta^{g(0)} + t\theta^{g(1)} + t^2\theta^{g(2)} + t^3\theta^{g(3)} + t^4\theta^{g(4)} + \cdots\right]$$

$$-\left[\theta^{g(1)} + t\theta^{g(2)} + t^2\theta^{g(3)} + t^3\theta^{g(4)} + t^4\theta^{g(5)} + \cdots\right]$$

$$= 1 + \theta^{g(1)}(t-1) + \theta^{g(2)}t(t-1) + \theta^{g(3)}t^2(t-1) + \cdots$$

$$= 1 + (t-1)\sum_{x=1}^{\infty} t^{(x-1)}\, \theta^{g(x)} .$$

4.2.2 Estimation of Parameters of the Discretized General Class

The parameters of the three distributions DMWE (α, β, θ), DMW (I) (α, β, θ), and DMW (II) (α, β, θ) will be estimated by the Proportion method, the Moments method, and the maximum likelihood method. For each method, the parameters will be estimated in four cases..

(1) The Proportion Method

$P(x;\alpha,\beta,\theta)=\theta^{g(x)}-\theta^{g(x+1)}$, where $g(x)=g(x;\alpha,\beta)$

Let y be the number of the zero's in the observed sample, and put

$P(0;\alpha,\beta,\theta)=\theta^{g(0)}-\theta^{g(1)}=\frac{y}{n}$, since $g(0)=0$

$$\xrightarrow{yields} 1-\theta^{g(1)}=\frac{y}{n} \tag{4.13}$$

Let z be the number of the one's in the observed sample, and put

$$P(1;\alpha,\beta,\theta)=\theta^{g(1)}-\theta^{g(2)}=\frac{z}{n} \tag{4.14}$$

Let w be the number of the two's in the observed sample, and put

$$P(2;\alpha,\beta,\theta)=\theta^{g(2)}-\theta^{g(3)}=\frac{w}{n} \tag{4.15}$$

This system of nonlinear equations is solved analytically.

(2) The Moments Method

Equating the sampling moments to the population moments, we can obtain the following system of nonlinear equations

$$\sum_{i=0}^{\infty} {x_i}^r\left(\theta^{g(x_i)}-\theta^{g(x_i+1)}\right)=\frac{1}{n}\sum_{i=1}^{n} {x_i}^r, \text{ for } r = 1, 2, 3. \tag{4.16}$$

This system also is solved analytically.

(3) The Maximum likelihood Method

The likelihood function L and the log likelihood function ln L are respectively, $L=\prod_{i=1}^{n}\left(\theta^{g(x_i)}-\theta^{g(x_i+1)}\right)$,

$\ln L = \sum_{i=1}^{n} \ln\left(\theta^{g(x_i)} - \theta^{g(x_i+1)}\right).$

We have to solve the following system

$$\left.\begin{array}{l} \frac{\partial \ln L}{\partial \alpha}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \\ \frac{\partial \ln L}{\partial \beta}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \\ \frac{\partial \ln L}{\partial \theta}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \end{array}\right\} \tag{4.17}$$

We solved this system analytically.

4.3 Discrete Modified Weibull Extension Distribution

In the present section, the first discretizing method introduced in Section 3.2 of Chapter III, will be applied to the modified Weibull distribution. That is

$$g(x) = \left(e^{(\frac{x}{\alpha})^{\beta}} - 1\right). \tag{4.18}$$

The survival function of the MWE$(\alpha, \beta, \lambda_1)$ distribution in the continuous case is

$$\begin{aligned} S(x) &= e^{\lambda_1 \alpha\left(1 - e^{(\frac{x}{\alpha})^{\beta}}\right)} \\ &= e^{-\lambda_1 \alpha\left(e^{(\frac{x}{\alpha})^{\beta}} - 1\right)} \\ &= \theta^{\left(e^{(\frac{x}{\alpha})^{\beta}} - 1\right)}, \end{aligned} \tag{4.19}$$

Here, $\theta = e^{-\lambda_1 \alpha}, 0 < \theta < 1$, by using Equation (4.3).

The probability mass function of the DMWE (α, β, θ) distribution is

$$P(x) = \theta^{-1}\left[\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{x+1}{\alpha}\right)^{\beta}}}\right], \quad x = 0,1,2, \dots \tag{4.20}$$

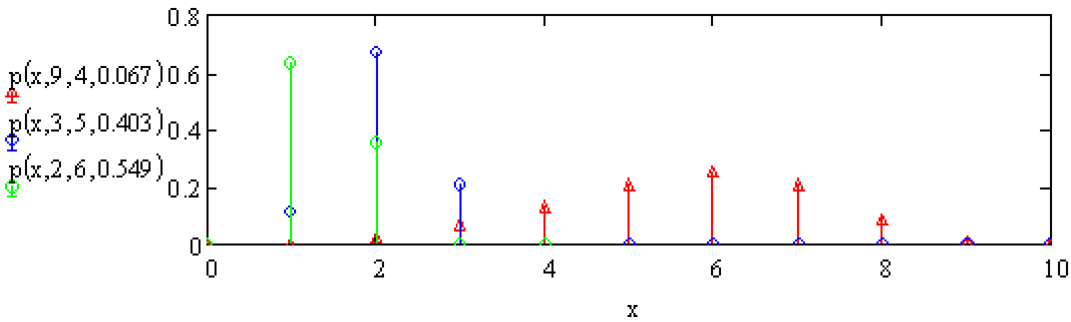


Figure 4.1 The pmf of DMWE distribution at different parameters' values.

To prove that $P(x)$ is a probability mass function it should be

(i) $P(x) \geq 0\,,\ trivial\,, since\ \alpha > 0, \beta > 0\,, 0 < \theta < 1$

(ii) $\sum_{x=0}^{\infty} P(x) = 1$.

Proof:

$$\sum_{x=0}^{\infty} P(x) = \sum_{x=0}^{\infty} \theta^{-1}[\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{x+1}{\alpha}\right)^{\beta}}}]$$

$$= \theta^{-1} \sum_{x=0}^{\infty} [\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{x+1}{\alpha}\right)^{\beta}}}]$$

$$= \theta^{-1}\{\left[\theta^{e^{\left(\frac{0}{\alpha}\right)^{\beta}}} + \theta^{e^{\left(\frac{1}{\alpha}\right)^{\beta}}} + \theta^{e^{\left(\frac{2}{\alpha}\right)^{\beta}}} + \cdots.\right]$$

$$-\left[\theta^{e^{\left(\frac{1}{\alpha}\right)^{\beta}}} + \theta^{e^{\left(\frac{2}{\alpha}\right)^{\beta}}} + \theta^{e^{\left(\frac{3}{\alpha}\right)^{\beta}}} + \cdots.\right]\}$$

$$= \theta^{-1}\theta^{e^{\left(\frac{0}{\alpha}\right)^{\beta}}}$$

$$= \theta^{-1}\theta = 1.$$

Substituting Equation (4.18) into Equation (4.4) yields the cumulative distribution function of the DMWE(α, β, θ) distribution in the form

$$F_1(x) = 1 - \theta^{e^{\left(\frac{x+1}{\alpha}\right)^{\beta}} - 1}. \tag{4.21}$$

Thus $F_1(x)$ equals the cdf $F(x)$ of the MWE $(\alpha, \beta, \lambda_1)$ distribution calculated at (x+1) in the continuous case. That is, $F_1(x) = F(x+1)$.

$Lim_{x\to 0} F_1(x) = 0$. Since, for large $\alpha, \beta > 1, \left(\frac{1}{\alpha}\right)^{\beta}$ be small this implies $e^{\left(\frac{1}{\alpha}\right)^{\beta}}$ is close to one then $e^{\lambda\alpha[1-1]}$ is close to one .

$Lim_{x\to\infty} F_1(x) \; = 1$ Since, $\theta^{e^{\left(\frac{\infty}{\alpha}\right)^{\beta}}} = e^{-\infty} = 0.$

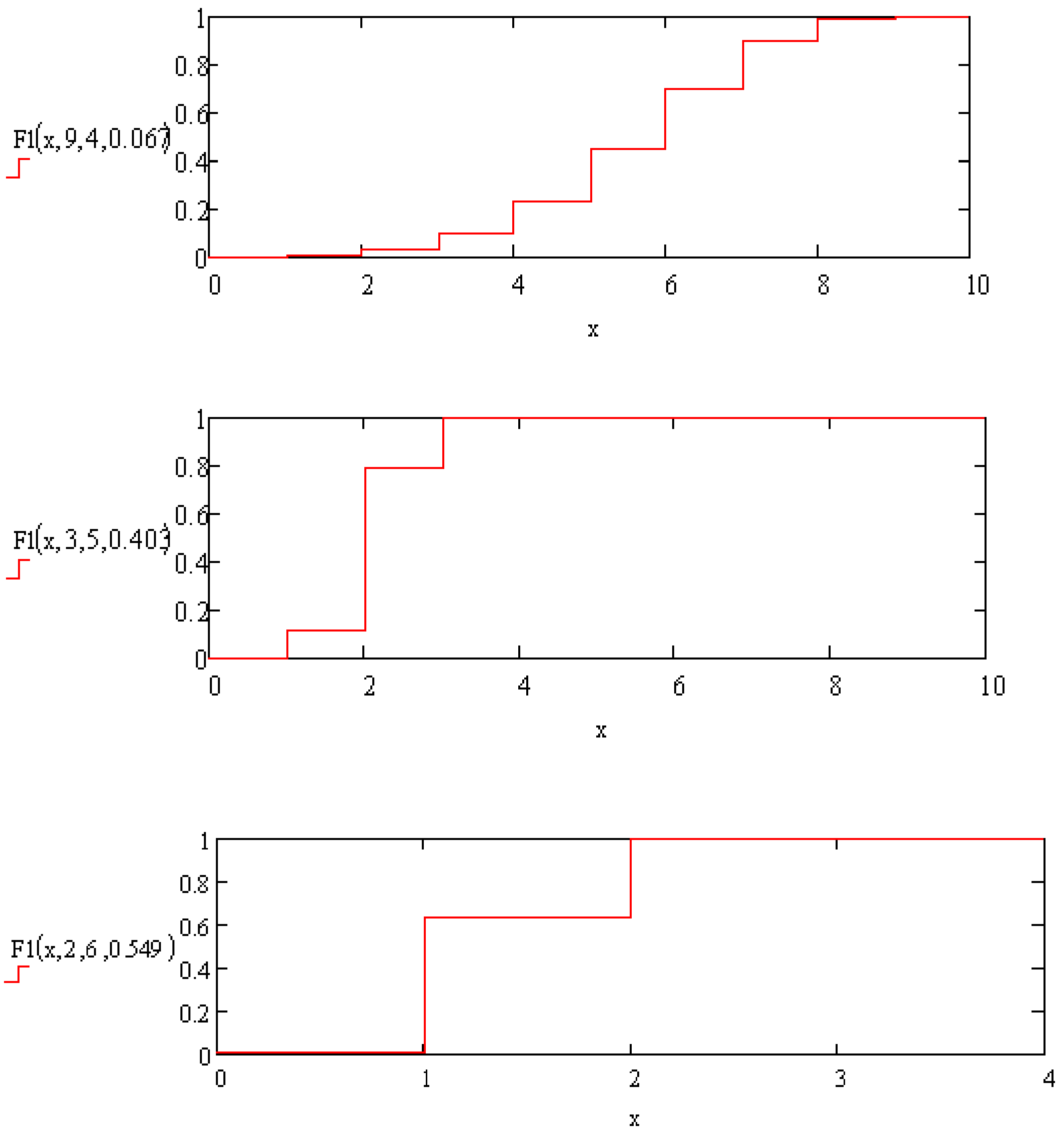


Figure 4.2 The cdf of DMWE distributions at different parameters' values

4.3.1 Some Reliability Measures for the DMWE Distribution

Substituting Equation (4.18) into Equation (4.5) yields the survival function of the DMWE (α, β, θ) distribution in the form

$$S_1(x) = \theta^{e^{\left(\frac{x+1}{\alpha}\right)^{\beta}}-1}. \tag{4.22}$$

This is the same as the survival function in the continuous case but calculated at $(x+1)$.

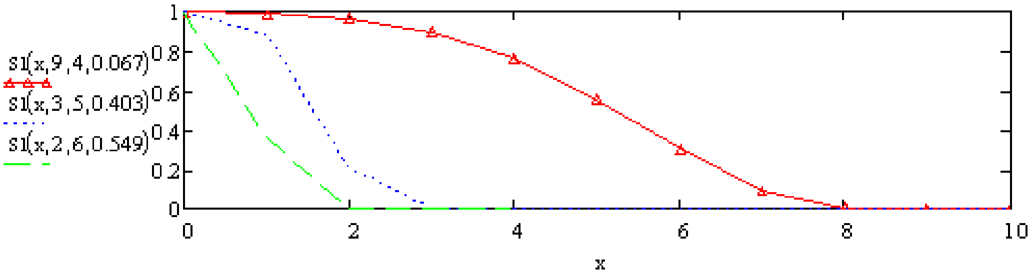


Figure 4.3 The survival function of DMWE distributions at different parameters' values

Substituting Equation (4.18) into Equation (4.6) yields the failure rate function of the DMWE(α, β, θ) distribution in the form

$$r_1(x) = 1 - \theta^{e^{\left(\frac{x+1}{\alpha}\right)^{\beta}} - e^{\left(\frac{x}{\alpha}\right)^{\beta}}}. \tag{4.23}$$

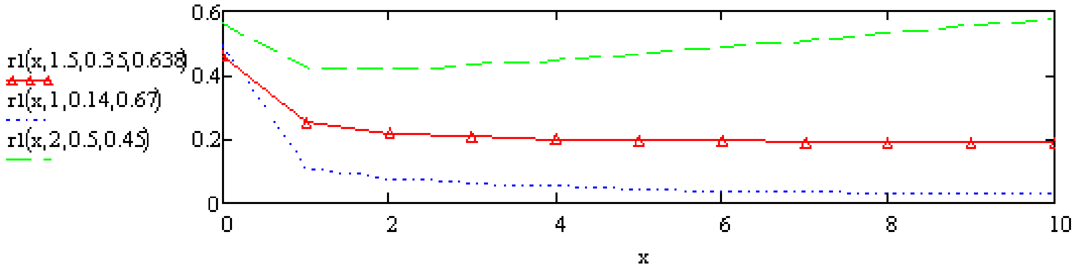


Figure 4.4 The failure rate function of DMWE distribution at different parameters' values

Substituting Equation (4.18) into Equation (4.7) yields the second failure rate function of the DMWE (α, β, θ) distribution in the form

$$SRF_1(x) = \left(e^{\left(\frac{x}{\alpha}\right)^{\beta}} - e^{\left(\frac{x+1}{\alpha}\right)^{\beta}}\right) ln\theta. \tag{4.24}$$

Substituting Equation (4.18) into Equation (4.8) yields the residual reliability function of the DMWE(α, β, θ) distribution, defined for all $i \in N^*$ in the form

$$R_1(i|x) = \theta^{\left[e^{\left(\frac{x+1+i}{\alpha}\right)^{\beta}} - e^{\left(\frac{x+1}{\alpha}\right)^{\beta}}\right]}. \tag{4.25}$$

Substituting Equation (4.18) into Equation (4.9) yields the cumulative hazard function of the DMWE(α, β, θ) distribution in the form

$$H_1(x) = x - \sum_{t=1}^{x} \theta^{\left[exp\left[\left(\frac{t+1}{\alpha}\right)^{\beta}\right] - exp\left[\left(\frac{t}{\alpha}\right)^{\beta}\right]\right]}. \tag{4.26}$$

The DMWE(α, β, θ) distribution has an increasing failure rate (IFR), since the two equivalent conditions are satisfied

IFR1: $\left\{1 - \theta^{\left[exp\left[\left(\frac{x+1}{\alpha}\right)^{\beta}\right] - exp\left[\left(\frac{x}{\alpha}\right)^{\beta}\right]\right]}\right\}_{x \geq 1}$ is an increasing sequence.

IFR2: For all $i \in N^*$, $\left\{\theta^{\left[exp\left[\left(\frac{x+i+1}{\alpha}\right)^{\beta}\right] - exp\left[\left(\frac{x+1}{\alpha}\right)^{\beta}\right]\right]}\right\}_{x \geq 1}$ is a decreasing sequence.

The DMWE(α, β, θ) distribution has also an increasing failure rate in average (IFRA), since the two equivalent conditions are satisfied

IFRA1: $\left\{\left(\theta^{exp\left[\left(\frac{x+1}{\alpha}\right)^{\beta}\right] - 1}\right)^{1/x}\right\}_{x \geq 1}$ is a decreasing sequence.

IFRA2: $\left\{1 - \frac{1}{x} \sum_{t=1}^{x} \theta^{\left[exp\left[\left(\frac{t+1}{\alpha}\right)^{\beta}\right] - exp\left[\left(\frac{t}{\alpha}\right)^{\beta}\right]\right]}\right\}_{x \geq 1}$ is an increasing sequence.

4.3.2 Properties of the DMWE Distribution

Substituting Equation (4.18) into Equation (4.10) yields the r^{th} moments about zero of the DMWE(α, β, θ) distribution in the form

$$\mu_r' = E(X^r) = \theta^{-1} \sum_{x=1}^{\infty} (x^r - (x-1)^r) \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}. \tag{4.27}$$

The first four moments about zero follows from Equation (4.27) in the form

$$\mu_1' = E(X) = \theta^{-1} \sum_{x=1}^{\infty} \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}},$$

$$\mu_2' = E(X^2) = \theta^{-1} \sum_{x=1}^{\infty} (2x-1) \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}},$$

$$\mu_3' = E(X^3) = \theta^{-1} \sum_{x=1}^{\infty} (3x^2 - 3x + 1) \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}},$$

$$\mu_4' = E(X^4) = \theta^{-1} \sum_{x=1}^{\infty} (4x^3 - 6x^2 + 4x - 1) \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}.$$

When $\beta = 1$, the above four formulae become

$$\left.\begin{aligned} \mu_1' &= E(X) = \theta^{-1} \sum_{x=1}^{\infty} \theta^{e^{\left(\frac{x}{\alpha}\right)}} \\ \mu_2' &= E(X^2) = \theta^{-1} \sum_{x=1}^{\infty} (2x-1) \theta^{e^{\left(\frac{x}{\alpha}\right)}} \\ \mu_3' &= E(X^3) = \theta^{-1} \sum_{x=1}^{\infty} (3x^2 - 3x + 1) \theta^{e^{\left(\frac{x}{\alpha}\right)}} \\ \mu_4' &= E(X^4) = \theta^{-1} \sum_{x=1}^{\infty} (4x^3 - 6x^2 + 4x - 1) \theta^{e^{\left(\frac{x}{\alpha}\right)}} \end{aligned}\right\} \tag{4.28}$$

The median of the DMWE(α, β, θ) distribution is

$$m = \alpha \left(\ln \left(1 - \frac{\ln 2}{\ln \theta} \right) \right)^{1/\beta} - 1. \tag{4.29}$$

Proof:

$$P(X \le m) \ge \frac{1}{2} \xrightarrow{yields} \left[1 - \theta^{-1} \theta^{e^{\left(\frac{m+1}{\alpha}\right)^{\beta}}} \right] \ge \frac{1}{2}$$

$$\xrightarrow{yields} -\theta^{-1} \theta^{e^{\left(\frac{m+1}{\alpha}\right)^{\beta}}} \ge -\frac{1}{2}$$

$$\xrightarrow{yields} \theta^{-1} \theta^{e^{\left(\frac{m+1}{\alpha}\right)^{\beta}}} \le \frac{1}{2}$$

$$\xrightarrow{yields} \theta^{e^{\left(\frac{m+1}{\alpha}\right)^{\beta}}} \le \frac{\theta}{2}$$

$$\xrightarrow{yields} e^{\left(\frac{m+1}{\alpha}\right)^{\beta}} ln\theta \leq \ln \frac{\theta}{2}$$

$$\xrightarrow{yields} e^{\left(\frac{m+1}{\alpha}\right)^{\beta}} \leq \left(\frac{\ln\theta - \ln 2}{ln\theta}\right)$$

$$\xrightarrow{yields} e^{\left(\frac{m+1}{\alpha}\right)^{\beta}} \leq 1 - \frac{\ln 2}{\ln\theta}$$

$$\xrightarrow{yields} \left(\frac{m+1}{\alpha}\right)^{\beta} \leq \ln\left(1 - \frac{\ln 2}{\ln\theta}\right)$$

$$\xrightarrow{yields} \frac{m+1}{\alpha} \leq \left(\ln\left(1 - \frac{\ln 2}{\ln\theta}\right)\right)^{1/\beta}$$

$$\xrightarrow{yields} m \leq \alpha\left(\ln\left(1 - \frac{\ln 2}{\ln\theta}\right)\right)^{1/\beta} - 1.$$

Similarly, for $P(X \geq m) \geq \frac{1}{2}$ we obtain $m \leq \alpha\left(\ln\left(1 - \frac{\ln 2}{\ln\theta}\right)\right)^{1/\beta} - 1.$

The mode of the DMWE (α, β, θ) distribution can be located graphically. The mode values corresponding to (α, β, θ) = (9, 4, 0.067), (3, 5, 0.403), (2, 6, 0.549) are, respectively D = 6, 2, 1. This is illustrated in Figure 4.1.

Substituting Equation (4.18) into Equation (4.11) yields the moment generating function of the DMWE (α, β, θ) distribution in the form

$$M_X(t) = E(e^{tX}) = 1 + \theta^{-1}\sum_{x=1}^{\infty}\left(e^{xt} - e^{(x-1)t}\right)\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}. \quad (4.30)$$

Differentiating Equation (4.30) *r* times with respect to *t*, we obtain

$$M_X^{(r)}(t) = \theta^{-1}\sum_{x=1}^{\infty}\left(x^r e^{xt} - (x-1)^r e^{(x-1)t}\right)\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}, r = 1,2,\ldots \quad (4.31)$$

The first four moments can be also obtained from Equation (4.31) when $t = 0$ and r = 1,2,3,4 in the form

$$\mu_1' = M_X'(t)|_{t=0} = \theta^{-1} \sum_{x=1}^{\infty} \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}},$$

$$\mu_2' = M_X''(t)|_{t=0} = \theta^{-1} \sum_{x=1}^{\infty} (2x-1)\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}},$$

$$\mu_3' = M_X^{(3)}(t)|_{t=0} = \theta^{-1} \sum_{x=1}^{\infty} (3x^2-3x+1)\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}},$$

$$\mu_4' = M_X^{(4)}(t)|_{t=0} = \theta^{-1} \sum_{x=1}^{\infty} (4x^3-6x^2+4x-1)\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}.$$

When $\beta = 1$, the above formulae become:

$$\mu_1' = M_X'(t)|_{t=0} = \theta^{-1} \sum_{x=1}^{\infty} \theta^{e^{\left(\frac{x}{\alpha}\right)}},$$

$$\mu_2' = M_X''(t)|_{t=0} = \theta^{-1} \sum_{x=1}^{\infty} (2x-1)\theta^{e^{\left(\frac{x}{\alpha}\right)}},$$

$$\mu_3' = M_X^{(3)}(t)|_{t=0} = \theta^{-1} \sum_{x=1}^{\infty} (3x^2-3x+1)\theta^{e^{\left(\frac{x}{\alpha}\right)}},$$

$$\mu_4' = M_X^{(4)}(t)|_{t=0} = \theta^{-1} \sum_{x=1}^{\infty} (4x^3-6x^2+4x-1)\theta^{e^{\left(\frac{x}{\alpha}\right)}}.$$

Substituting Equation (4.18) into Equation (4.12) yields the probability generating function of the DMWE (α, β, θ) distribution in the form

$$G_{[X]}(t) = E(t^X) = 1 + (t-1)\theta^{-1} \sum_{x=1}^{\infty} t^{(x-1)}\, \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}. \tag{4.32}$$

Differentiation of the both sides of Equation (4.32) with respect to *t*, gives the first and second derivatives in the form

$$G'_{[X]}(t) = \theta^{-1} \sum_{x=1}^{\infty} t^{(x-2)}(xt-(x-1))\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}. \tag{4.33}$$

$$G''_{[X]}(t) = \theta^{-1} \sum_{x=1}^{\infty} t^{(x-3)}(x-1)[xt-(x-1)]\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}. \tag{4.34}$$

Substituting *t*=1 into Equations (4.33) and (4.34) gives the first and second factorial moments in the form

$$\mu_{[1]} = E(X) = G'_{[X]}(1) = \theta^{-1} \sum_{x=1}^{\infty} \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}},$$

$$\mu_{[2]} = E\big(X(X-1)\big) = G''_{[X]}(1) = \theta^{-1} \sum_{x=1}^{\infty} 2(x-1)\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}.$$

The second moment and the variance of X are

$$E(X^2) = \mu_{[1]} + \mu_{[2]} = \theta^{-1} \sum_{x=1}^{\infty} (2x-1)\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}. \tag{4.35}$$

$$Var(X) = \theta^{-1} \sum_{x=1}^{\infty} (2x-1)\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}} - \left(\theta^{-1} \sum_{x=1}^{\infty} \theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}}\right)^2. \tag{4.36}$$

4.3.3 Estimation of the Parameters of the DMWE Distribution

The parameters of DMWE (α, β, θ) distribution will be estimated by the proportion method, the moments method and the maximum likelihood method.

(1) The Proportion Method

Case I: known parameters α and β and unknown parameter θ.

The unknown parameter θ has a proportion estimator in exact solution, where

$$\theta^* = \left(1 - \frac{y}{n}\right)^{\left(e^{\left(\frac{1}{\alpha}\right)^{\beta}} - 1\right)^{-1}}. \tag{4.37}$$

Proof: Let y be the number of the zero's in the sample

$$P(0; \alpha, \beta, \theta) = \theta^{-1}\left[\theta^{e^{\left(\frac{0}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{1}{\alpha}\right)^{\beta}}}\right] = \frac{y}{n}$$

$$\xrightarrow{yields} \theta^{-1}\left[\theta - \theta^{e^{\left(\frac{1}{\alpha}\right)^{\beta}}}\right] = \frac{y}{n} \tag{4.38}$$

$$\xrightarrow{yields} \left[1 - \theta^{e^{\left(\frac{1}{\alpha}\right)^{\beta}} - 1}\right] = \frac{y}{n}$$

$$\xrightarrow{yields} \theta^{e^{\left(\frac{1}{\alpha}\right)^{\beta}}-1} = 1 - \frac{y}{n}$$

$$\xrightarrow{yields} \left(e^{\left(\frac{1}{\alpha}\right)^{\beta}} - 1\right) \ln\theta = \ln\left(1 - \frac{y}{n}\right)$$

$$\xrightarrow{yields} \ln\theta = \frac{\ln\left(1-\frac{y}{n}\right)}{\left(e^{\left(\frac{1}{\alpha}\right)^{\beta}}-1\right)}$$

$$\xrightarrow{yields} \theta^{*} = e^{\left(\frac{\ln\left(1-\frac{y}{n}\right)}{\left(e^{\left(\frac{1}{\alpha}\right)^{\beta}}-1\right)}\right)}$$

$$\xrightarrow{yields} \theta^{*} = \left(1 - \frac{y}{n}\right)^{\left(e^{\left(\frac{1}{\alpha}\right)^{\beta}}-1\right)^{-1}}.$$

Case II: known parameter β and unknown parameters α and θ.

Let z be the number of the one's in the sample

$$\theta^{-1}\left[\theta^{e^{\left(\frac{1}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{2}{\alpha}\right)^{\beta}}}\right] = \frac{z}{n}$$

$$\xrightarrow{yields} \theta^{e^{\left(\frac{1}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{2}{\alpha}\right)^{\beta}}} = \theta\,\frac{z}{n}. \tag{4.39}$$

Solving Equations (4.38) and (4.39) numerically gives the proportion estimators α^{*} and θ^{*} of the parameters α and θ.

Case III: known Parameter α and unknown parameters β and θ.

Solving Equations (4.38) and (4.39) numerically gives the proportion estimators β^{*}, θ^{*} of the parameters β and θ.

Case IV: unknown parameters α, β, θ .

Let w be the number of the two's in the sample

$$P(2;\alpha,\theta,\beta) = \theta^{-1}\left[\theta^{e^{\left(\frac{2}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{3}{\alpha}\right)^{\beta}}}\right] = \frac{w}{n}$$

$$\xrightarrow{yields} \theta^{e^{\left(\frac{2}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{3}{\alpha}\right)^{\beta}}} = \theta\,\frac{w}{n}. \tag{4.40}$$

Solving the Equations (4.38)-(4.40) numerically, gives the proportion estimators $\alpha^*, \beta^*, \theta^*$ of the parameters α, β, θ.

(2) The Moments Method

Case I: known parameters α and β and unknown parameter θ.

Equating the first population moment to the first sample moment gives the equation

$$\sum_{x=0}^{\infty} x\,\theta^{-1}\left[\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{x+1}{\alpha}\right)^{\beta}}}\right] = \frac{1}{n}\sum_{i=1}^{n} x_i\ . \tag{4.41}$$

Solving the Equation (4.41) numerically gives the method-of-moments estimator $\tilde{\theta}$ of the parameter θ.

Case II: known parameter β and unknown parameters α and .

Equating the second population moment to the second sample moment gives

$$\sum_{x=0}^{\infty} x^2\,\theta^{-1}\left[\theta^{e^{\left(\frac{x}{\alpha}\right)^{\beta}}} - \theta^{e^{\left(\frac{x+1}{\alpha}\right)^{\beta}}}\right] = \frac{1}{n}\sum_{i=1}^{n} x_i^2\ . \tag{4.42}$$

Solving the Equations (4.41) and (4.42) numerically gives the method-of-moments estimators $\tilde{\alpha}$ and $\tilde{\theta}$ of the parameters α, θ.

Case III: known parameter α and unknown parameters β and θ.

Solving the Equations (4.41) and (4.42) numerically gives the method-of-moments estimators $\tilde{\beta}$ and $\tilde{\theta}$ of the parameters β, θ.

Case IV: unknown parameters α, β, θ.

Equating the third population moment to the third sample moment gives

$$\sum_{x=0}^{\infty} x^3\, \theta^{-1}\left[\theta^{e^{\left(\frac{x}{\alpha}\right)^\beta}} - \theta^{e^{\left(\frac{x+1}{\alpha}\right)^\beta}}\right] = \frac{1}{n}\sum_{i=1}^{n} x_i^3. \qquad (4.43)$$

Solving the Equations (4.41) - (4.43) numerically gives the method-of-moments estimators $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}$ of the parameters α, β, θ.

(3) The Maximum likelihood Method

The likelihood function and the log likelihood function of the DMWE (α, β, θ) distribution are

$$L = \theta^{-n} \prod_{i=1}^{n}\left[\theta^{e^{\left(\frac{x_i}{\alpha}\right)^\beta}} - \theta^{e^{\left(\frac{x_i+1}{\alpha}\right)^\beta}}\right],$$

$$\ln L = -n\, ln\, \theta + \sum_{i=1}^{n} ln\left[\theta^{e^{\left(\frac{x_i}{\alpha}\right)^\beta}} - \theta^{e^{\left(\frac{x_i+1}{\alpha}\right)^\beta}}\right].$$

Case I: known parameters $\alpha\ and\ \beta$ and unknown parameter θ.

$$\frac{\partial\, \ln L}{\partial \theta}\Big|_{\alpha=\hat{\alpha}, \beta=\hat{\beta}, \theta=\hat{\theta}} = 0 \xrightarrow{\text{yields}}$$

$$\frac{-n}{\theta} + \sum_{i=1}^{n} \frac{e^{\left(\frac{x_i}{\alpha}\right)^\beta} . \theta^{\left(e^{\left(\frac{x_i}{\alpha}\right)^\beta} - 1\right)} - e^{\left(\frac{x_i+1}{\alpha}\right)^\beta} \theta^{\left(e^{\left(\frac{x_i+1}{\alpha}\right)^\beta} - 1\right)}}{\theta^{e^{\left(\frac{x_i}{\alpha}\right)^\beta}} - \theta^{e^{\left(\frac{x_i+1}{\alpha}\right)^\beta}}} = 0. \qquad (4.44)$$

Solving the Equation (4.44) analytically gives the maximum likelihood estimator $\hat{\theta}$ of the parameter θ.

Case II: known parameter β is and unknown parameters α and θ.

$$\frac{\partial \ln L}{\partial \alpha}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{\theta^{e^{\left(\frac{x_i}{\alpha}\right)^\beta}} e^{\left(\frac{x_i}{\alpha}\right)^\beta} . \left(\frac{x_i}{\alpha}\right)^{\beta-1} - \theta^{e^{\left(\frac{x_i+1}{\alpha}\right)^\beta}} e^{\left(\frac{x_i+1}{\alpha}\right)^\beta} . \left(\frac{x_i+1}{\alpha}\right)^{\beta-1}}{\theta^{e^{\left(\frac{x_i}{\alpha}\right)^\beta}} - \theta^{e^{\left(\frac{x_i+1}{\alpha}\right)^\beta}}} = 0. \qquad (4.45)$$

Solving the Equations (4.44) and (4.45) analytically gives the maximum likelihood estimators $\hat{\alpha}, \hat{\theta}$ of the parameters α, θ.

Case III: known parameter α is and unknown parameters β and θ.

$$\frac{\partial \ln L}{\partial \beta}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{\theta^{e^{\left(\frac{x_i}{\alpha}\right)^\beta}} .e^{\left(\frac{x_i}{\alpha}\right)^\beta} . \left(\frac{x_i}{\alpha}\right)^{\beta} .ln\left(\frac{x_i}{\alpha}\right) - \theta^{e^{\left(\frac{x_i+1}{\alpha}\right)^\beta}} .e^{\left(\frac{x_i+1}{\alpha}\right)^\beta} . \left(\frac{x_i+1}{\alpha}\right)^{\beta} .ln\left(\frac{x_i+1}{\alpha}\right)}{\theta^{e^{\left(\frac{x_i}{\alpha}\right)^\beta}} - \theta^{e^{\left(\frac{x_i+1}{\alpha}\right)^\beta}}} = 0. \qquad (4.46)$$

Solving the Equations (4.44) and (4.46) analytically gives the maximum likelihood estimators $\hat{\beta}, \hat{\theta}$ of the parameters β, θ.

Case IV: unknown parameters α, β, θ.

Solving the Equations (4.44) - (4.46) analytically gives the maximum likelihood estimators $\hat{\alpha}, \hat{\beta}, \hat{\theta}$ of the parameters α, β, θ.

4.3.4 Special Distributions from DMWE Distribution

Many discretized distributions follow as special cases from DMWE distribution. Examples of such distributions are the discretized model of Chen (2000), discrete Weibull and discrete exponential distributions.

As one of our new results, the discretized model of Chen distribution handled in detail. The remaining existing discretized distributions referred to shortly.

The DMWE (θ, α, β) distribution reduces to the model by Chen (2000) when $\alpha = 1$.

The pmf of the discrete model of Chen is

$$P(x; \beta, \theta) = \theta^{-1}[\theta^{e^{(x)^{\beta}}} - \theta^{e^{(x+1)^{\beta}}}], \quad x = 0,1,2, \dots \tag{4.47}$$

$$g(x) = e^{x^{\beta}-1}. \tag{4.48}$$

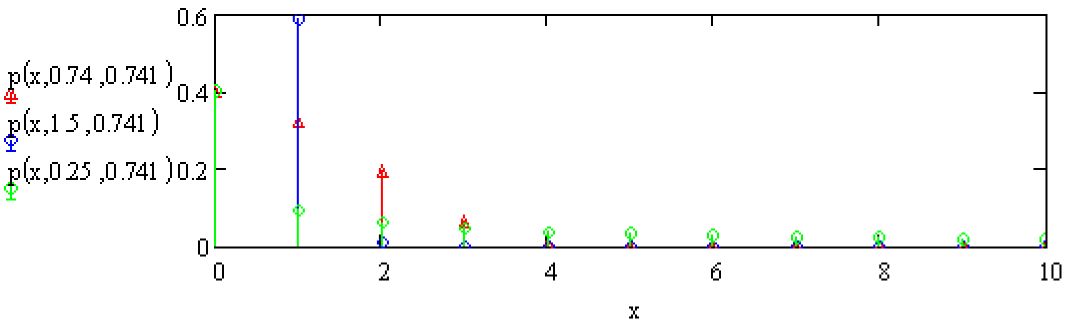
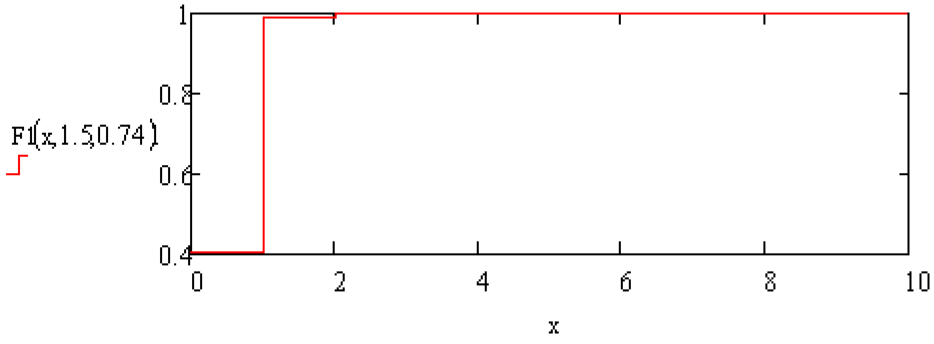
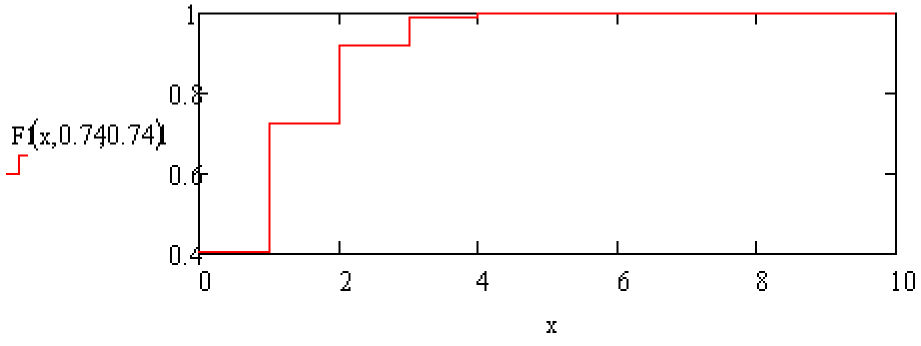


Figure 4.5 The pmf of DChen distribution at different parameters' values.

Substituting Equation (4.48) into Equation (4.4) yields the cumulative distribution of the DChen (β, θ) distribution in the form

$$F_1(x) = 1 - \theta^{e^{(x+1)^{\beta}}-1}.$$





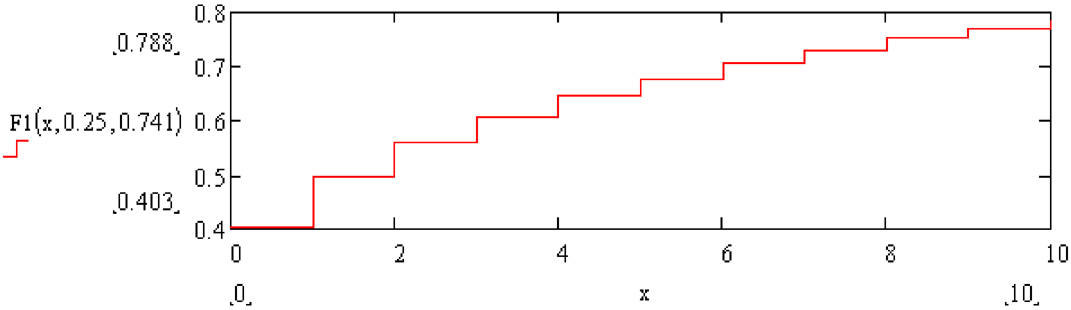


Figure 4.6 The cdf of DChen distributions at different parameters' values.

Substituting Equation (4.48) into Equation (4.5) yields the survival function of the DChen (β, θ) distribution in the form

$$S_1(x) = \theta^{e^{(x+1)^{\beta}}-1}.$$

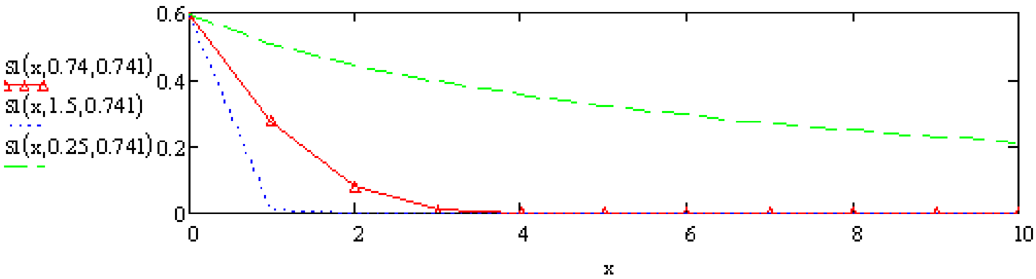


Figure 4.7 The survival function of DChen distributions at different parameters' values

Substituting Equation (4.48) into Equation (4.6) yields the failure rate function of the DChen (β, θ) distribution in the form

$$r_1(x) = 1 - \theta^{e^{(x+1)^{\beta}}-e^{x^{\beta}}}.$$

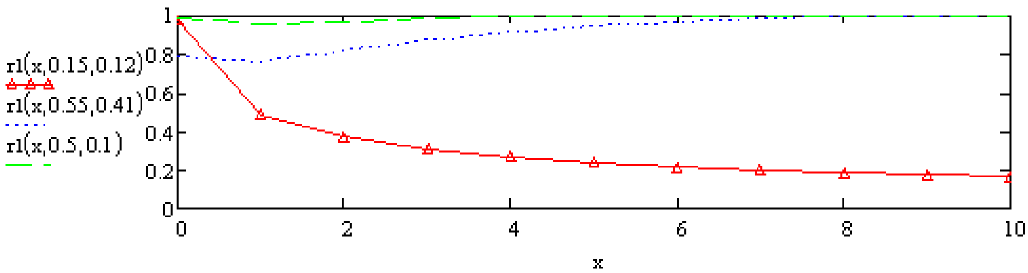


Figure 4.8 The failure rate function of DChen distribution at different parameters' values

Substituting Equation (4.48) into Equation (4.7) yields the second failure rate function of the DChen (β, θ) distribution in the form

$$SRF_1 = \left(e^{x^\beta} - e^{(x+1)^\beta}\right) ln\theta.$$

Substituting Equation (4.48) into Equation (4.8) yields the residual reliability function of the DChen (β, θ) distribution, defined for all $i \in N^*$ in the form

$$R_1(i|x) = \theta^{e^{(x+1+i)^\beta} - e^{(x+1)^\beta}}, x = 0,1, \ldots$$

Substituting Equation (4.48) into Equation (4.9) yields the cumulative hazard function of the DChen (β, θ) distribution in the form

$$H_1(x) = x - \sum_{t=1}^{x} \theta^{e^{(t+1)^\beta} - e^{t^\beta}}.$$

The DChen (β, θ) distribution has an increasing failure rate (IFR), since the following equivalent conditions are satisfied

IFR1: $\left\{1 - \theta^{e^{(x+1)^\beta} - e^{x^\beta}}\right\}_{x \geq 1}$ is an increasing sequence.

IFR2: For all $i \in N^*$ $\left\{\theta^{e^{(x+1+i)^\beta} - e^{(x+1)^\beta}}\right\}_{x \geq 1}$ is a decreasing sequence.

The DChen (β, θ) distribution has an increasing failure rate in average (IFRA), since the following equivalent conditions are satisfied

IFRA1: $\left\{\left(\theta^{e^{(x+1)^\beta} - 1}\right)^{1/x}\right\}_{x \geq 1}$ is a decreasing sequence.

IFRA2: $\left\{1 - \sum_{t=1}^{x} \theta^{e^{(t+1)^\beta} - e^{t^\beta}}\right\}_{x \geq 1}$ is an increasing sequence.

Substituting Equation (4.48) into Equation (4.10) yields the r^{th} moment about zero of the DChen (β, θ) distribution in the form

$$\mu_r' = E(X^r) = \sum_{x=1}^{\infty}(x^r - (x-1)^r)\theta^{e^{x^\beta}-1}.$$

The first four moments about zero are, respectively

$$\mu = \theta^{-1}\sum_{x=1}^{\infty}\theta^{e^{x^\beta}},$$

$$\mu_2' = \theta^{-1}\sum_{x=1}^{\infty}(2x-1)\theta^{e^{x^\beta}},$$

$$\mu_3' = \theta^{-1}\sum_{x=1}^{\infty}(3x^2-3x+1)\theta^{e^{x^\beta}},$$

$$\mu_4' = \theta^{-1}\sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\theta^{e^{x^\beta}}.$$

The median of the DChen (β, θ) distribution is obtained from Equation (4.30) when $\alpha = 1$, in the form

$$m = \left(\ln\left(1 - \frac{\ln 2}{\ln \theta}\right)\right)^{1/\beta} - 1.$$

The mode of the DChen (β, θ) distribution can be obtained graphically by plotting the pmf for different sets of values of the parameters. Figure 4.5 shows the mode D = 0, 1, 0. When (β, θ) = (0.74, 0.741), (1.5, 0.741), (0.25, 0.741), respectively.

Substituting Equation (4.48) into Equation (4.11) yields the moment generating function of the DChen (β, θ) distribution in the form

$$M_X(t) = E(e^{tX}) = 1 + \theta^{-1}\sum_{x=1}^{\infty}\left(e^{xt} - e^{(x-1)t}\right)\theta^{e^{x^\beta}}.$$

Differentiating the mgf $M_x(t)$ *r* times with respect to *t* gives the r^{th} moment of the DChen (β, θ) distribution in the form

$$\mu_r' = M_X^{(r)}(t)|_{t=0} = \theta^{-1}\sum_{x=1}^{\infty}\left(x^r e^{xt} - (x-1)^r e^{(x-1)t}\right)\theta^{e^{x^\beta}}.$$

From this equation, we can obtain the first four moments of the DChen (β,θ) distribution by simply substituting the values r = 1, 2, 3, 4.

$$\mu_1' = M_X'(t)|_{t=0} = \theta^{-1}\sum_{x=1}^{\infty}\theta^{e^{x\beta}} ,$$

$$\mu_2' = M_X''(t)|_{t=0} = \theta^{-1}\sum_{x=1}^{\infty}(2x-1)\theta^{e^{x\beta}} ,$$

$$\mu_3' = M_X^{(3)}(t)|_{t=0} = \theta^{-1}\sum_{x=1}^{\infty}(3x^2-3x+1)\theta^{e^{x\beta}},$$

$$\mu_4' = M_X^{(4)}(t)|_{t=0} = \theta^{-1}\sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\theta^{e^{x\beta}}.$$

Substituting Equation (4.48) into Equation (4.12) yields the probability generating function of the DChen (β,θ) distribution in the form

$$G_{[X]}(t) = E(t^X) = 1 + (t-1)\theta^{-1}\sum_{x=1}^{\infty}t^{(x-1)}\,\theta^{e^{x\beta}}.$$

Differentiation of the both sides of the above equation with respect to t, gives the first and second derivatives in the form

$$G'_{[X]}(t) = \theta^{-1}\sum_{x=1}^{\infty}t^{(x-2)}(xt-x+1)\theta^{e^{x\beta}}.$$

$$G''_{[X]}(t) = \theta^{-1}\sum_{x=1}^{\infty}(x-1)t^{(x-3)}(xt-x+2)\theta^{e^{x\beta}}.$$

At t=1, these equations yields the first and second factorial moments in the form

$$\mu_{[1]} = G'_{[X]}(1) = \theta^{-1}\sum_{x=1}^{\infty}\theta^{e^{x\beta}} .$$

$$\mu_{[2]} = G''_{[X]}(1) = 2\theta^{-1}\sum_{x=1}^{\infty}(x-1)\theta^{e^{x\beta}}.$$

The mean, second moment and the variance of the DChen (β,θ) distribution are

$$\mu = \theta^{-1}\sum_{x=1}^{\infty}\theta^{e^{x\beta}},$$

$E(X^2) = \theta^{-1} \sum_{x=1}^{\infty} (2x - 1) \theta^{e^{x^\beta}}$,

$Var(X) = \theta^{-1} \sum_{x=1}^{\infty} (2x - 1) \theta^{e^{x^\beta}} - \left(\theta^{-1} \sum_{x=1}^{\infty} \theta^{e^{x^\beta}}\right)^2$.

The parameters of the DChen (β, θ) distribution will be estimated by:

(1) The Proportion Method

Case I: known parameter β and unknown parameter θ.

For known parameter β, the parameter θ will have the proportion estimator

$$\theta^* = \left(1 - \frac{y}{n}\right)^{(e-1)^{-1}}. \tag{4.49}$$

Proof: Let y be the number of the zero's in the sample

$$P(0; \theta, \beta) = \theta^{-1} \left[\theta^{e^{(0)^\beta}} - \theta^{e^{(1)^\beta}}\right] = \frac{y}{n}$$

$$\xrightarrow{yields} \theta^{-1}\left[\theta - \theta^{e^1}\right] = \frac{y}{n}$$

$$\xrightarrow{yields} 1 - \theta^{e-1} = \frac{y}{n}. \tag{4.50}$$

Solving Equation (4.50) in θ gives the proportion estimator θ^* in the form

$$\xrightarrow{yields} \theta^* = e^{\left(\frac{\ln\left(1 - \frac{y}{n}\right)}{(e-1)}\right)} = \left(1 - \frac{y}{n}\right)^{(e-1)^{-1}}.$$

Case II: unknown parameters $and\ \theta$.

Let z be the number of the one's in the sample

$$P(1; \theta, \beta) = \theta^{-1} \left[\theta^{e^{(1)^\beta}} - \theta^{e^{(2)^\beta}}\right] = \frac{z}{n}$$

$$\xrightarrow{yields} \theta^{-1} \left[\theta^{e} - \theta^{e^{(2)^\beta}}\right] = \frac{z}{n}. \tag{4.51}$$

Solving the Equations (4.50) and (4.51) gives the proportion estimators β^*, θ^*.

(2) The Moments Method

Case I: known parameter β and unknown parameter θ.

Equating the first population moment to the first sample moment gives the equation

$$\theta^{-1} \sum_{x=0}^{\infty} \theta^{e^{x^{\beta}}} = \frac{1}{n} \sum_{i=1}^{n} x_i \, . \qquad (4.52)$$

Solving the Equation (4.52) numerically yields the moment estimator $\tilde{\theta}$ of the parameter θ.

Case II: unknown parameters β, θ.

Equating the second population moment to the second sample moment gives

$$\theta^{-1} \sum_{x=0}^{\infty} (2x-1) \theta^{e^{(x)^{\beta}}} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 . \qquad (4.53)$$

Solving the Equations (4.52) and (4.53) gives the moments estimators $\tilde{\beta}, \tilde{\theta}$ of the parameters β, θ.

(3) The Maximum likelihood Method

The likelihood and the log likelihood functions of the DChen (β, θ) distribution are,

$$L = \prod_{i=1}^{n} \theta^{-1} \left(\theta^{e^{(x_i)^{\beta}}} - \theta^{e^{(x_i+1)^{\beta}}} \right),$$

$$\ln L = -n \ln \theta + \sum_{i=1}^{n} \ln \left(\theta^{e^{(x_i)^{\beta}}} - \theta^{e^{(x_i+1)^{\beta}}} \right).$$

Case I: known parameter β and unknown parameter .

$$\frac{\partial \ln \mathrm{L}}{\partial \theta} \Big|_{\beta=\hat{\beta}, \theta=\hat{\theta}} = 0 \xrightarrow{\text{yields}}$$

$$\frac{-n}{\theta}+\sum_{i=1}^{n}\frac{e^{(x_i)^{\beta}}.\theta^{\left(e^{(x_i)^{\beta}}-1\right)}-e^{(x_i+1)^{\beta}}\theta^{(e^{(x_i+1)^{\beta}}-1)}}{\theta^{e^{(x_i)^{\beta}}}-\theta^{e^{(x_i+1)^{\beta}}}}=0. \quad (4.54)$$

Solving the Equation (4.54) gives the maximum likelihood estimator $\hat{\theta}$ of the parameter θ.

Case II: unknown parameters θ,β.

$$\frac{\partial \ln L}{\partial \beta}|_{\beta=\hat{\beta},\theta=\hat{\theta}}=0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n}\frac{\theta^{e^{(x_i)^{\beta}}}.e^{(x_i)^{\beta}}.(x_i)^{\beta}.ln(x_i)-\theta^{e^{(x_i+1)^{\beta}}}.e^{(x_i+1)^{\beta}}.(x_i+1)^{\beta}.ln(x_i+1)}{\theta^{e^{(x_i)^{\beta}}}-\theta^{e^{(x_i+1)^{\beta}}}}=0. \quad (4.55)$$

Solving the Equations (4.54) and (4.55) analytically gives the maximum likelihood estimators $\hat{\beta},\hat{\theta}$ of the parameters β,θ.

Other special cases that result in other discretized probability distribution do exist. For example, the DMWE (α,β,θ) distribution reduces to the discrete Weibull DW (β,θ) distribution when the scale parameter α becomes very large or approaches infinity. This gives the survival function $S_1(x)=\theta^{(x+1)^{\beta}}$, which corresponds to the discrete Weibull distribution with two parameters β,θ.

(Nakagawa and Osaki (1975))

Another special case of interest follows when $\beta=1$ and the scale parameter α becomes very large or approaches infinity. In this case the DMWE (α,β,θ) distribution reduces to the discrete exponential distribution with parameter θ, with survival function $S_1(x)=\theta^{x+1},\theta=e^{-\lambda}$. This is referred as DE (θ) distribution.

4.4 Discrete Modified Weibull Type I Distribution

Applying the first discretizing method, introduced in Section 3.2 of Chapter III, to the general class of continuous distributions by putting

$$g(x) = \alpha x + x^{\beta}, \alpha = \frac{\alpha_1}{\lambda}, \tag{4.56}$$

into Equation (4.2) yields the survival function of the MW (I) $(\alpha_1, \beta, \lambda)$ distribution as

$$S(x) = \theta^{\alpha x + x^{\beta}}, \theta = e^{-\lambda}, \alpha = \frac{\alpha_1}{\lambda}. \tag{4.57}$$

By using Equation (4.3), so the pmf of the DMW (I) (α, β, θ) distribution is

$$P(x) = \theta^{\alpha x + x^{\beta}} - \theta^{\alpha(x+1)+(x+1)^{\beta}}, \quad x = 0,1,2, \dots \tag{4.58}$$

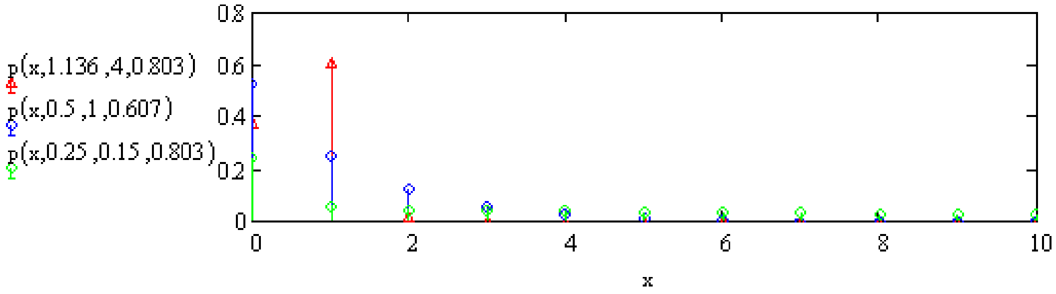


Figure 4.9 The pmf of DMW(I) distribution at different parameters' values.

To prove that $P(x)$ is a probability mass function it should be

(i)$P(x) \geq 0$, trivial, since $0 < \theta < 1, \alpha > 0$ and $\beta > 0$

(ii) $\sum_{x=0}^{\infty} P(x) = 1,$

Proof:

$$\sum_{x=0}^{\infty} P(x) = \sum_{x=0}^{\infty} \left[\theta^{\alpha x + x^{\beta}} - \theta^{\alpha(x+1)+(x+1)^{\beta}}\right]$$

$$= (\theta^{0} + \theta^{\alpha+1} + \theta^{\alpha+2} + \theta^{\alpha+3} + \cdots)$$

$$-(\theta^{\alpha+1} + \theta^{\alpha+2} + \theta^{\alpha+3} + \cdots)$$

$$= \theta^{0} = e^{-\lambda(0)} = 1.$$

Substituting Equation (4.56) into Equation (4.4) gives the cumulative distribution function of the DMW (I) (α, β, θ) distribution in the form

$$F_1(x) = 1 - \theta^{\alpha(x+1)+(x+1)^{\beta}}. \tag{4.59}$$

This is the cdf of the MW(I) (α, β, λ) distribution calculated at $(x+1)$ in the continuous case. That is $F_1(x) = F(x+1)$.

$$\lim_{x \to 0^+} F_1(x) = 1 - \theta^{\alpha(0)+(0)^{\beta}} = 1 - 1 = 0.$$

$$\lim_{x \to \infty^-} F_1(x) = 1 - \theta^{\infty} = 1 - e^{-\infty} = 1.$$

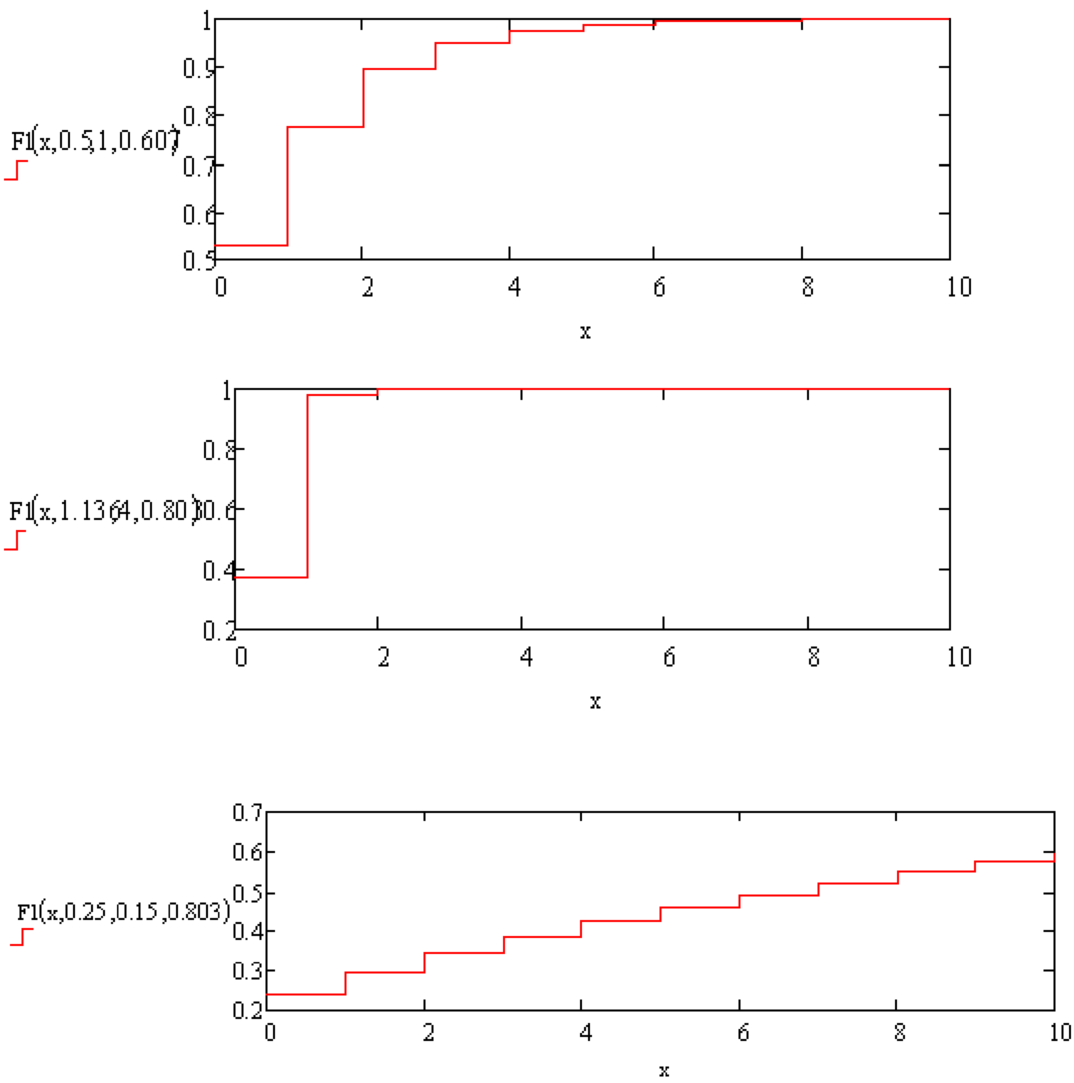


Figure 4.10 The cdf of DMW(I) distributions at different parameters' values.

4.4.1 Some Reliability Measures of the DMW (I) Distribution

Substituting Equation (4.56) into Equation (4.5) gives the survival function of the DMW (I) (α, β, θ) distribution in the form

$$S_1(x) = \theta^{\alpha(x+1)+(x+1)^{\beta}}. \tag{4.60}$$

which is the same as the survival function in the continuous case at $(x + 1)$.

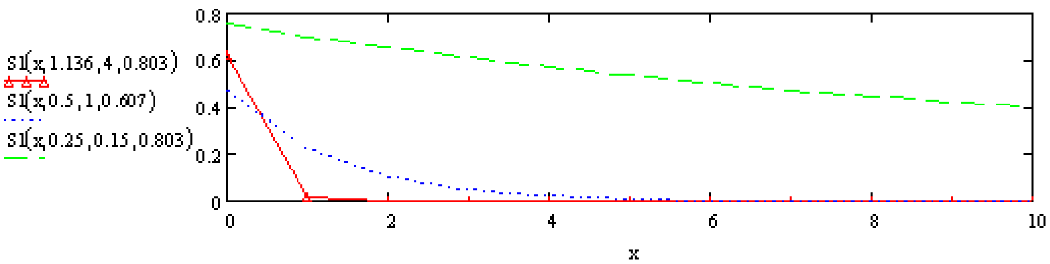


Figure 4.11 The survival function of DMW(I) distributions at different parameters' values

Substituting Equation (4.56) into Equation (4.6) gives the failure rate function of the DMW (I) (α, β, θ) distribution in the form

$$r_1(x) = 1 - \theta^{\alpha+(x+1)^{\beta}-x^{\beta}}. \tag{4.61}$$

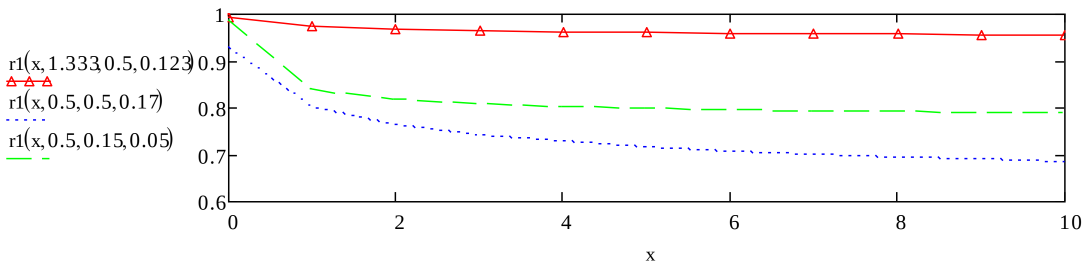


Figure 4.12 The failure rate function of DMW(I) distribution at different parameters' values

Substituting Equation (4.56) into Equation (4.7) gives the second failure rate function of the DMW (I) (α, β, θ) distribution in the form

$$SRF_1(x) = \left[x^{\beta} - \alpha - (x+1)^{\beta}\right] \ln \theta. \tag{4.62}$$

Substituting Equation (4.56) into Equation (4.8) gives the residual reliability function of the DMW (I) (α, β, θ) distribution in the form

$$R_1(i|x) = \theta^{i\alpha+(x+i+1)^\beta-(x+1)^\beta}. \tag{4.63}$$

Substituting Equation (4.56) into Equation (4.9) gives the cumulative hazard function of the DMW (I) (α, β, θ) distribution in the form

$$H_1(x) = x - \sum_{t=1}^{x} \theta^{\alpha+(t+1)^\beta-t^\beta}. \tag{4.64}$$

The DMW (I) (α, β, θ) distribution has an increasing failure rate (IFR), since the following equivalent conditions are satisfied

IFR1: $\left\{1-\theta^{\alpha+(x+1)^\beta-x^\beta}\right\}_{x\geq 1}$ is an increasing sequence.

IFR2: For all $i \in N^*$ $\left\{\theta^{i\alpha+(x+i+1)^\beta-(x+1)^\beta}\right\}_{x\geq 1}$ is a decreasing sequence.

Also, the DMW (I) (α, β, θ) distribution has an increasing failure rate in average (IFRA), since the following equivalent conditions are satisfied

IFRA1: $\left\{\left(\theta^{\alpha(x+1)+(x+1)^\beta}\right)^{1/x}\right\}_{x\geq 1}$ is a decreasing sequence.

IFRA2: $\left\{1-\frac{1}{x}\sum_{t=1}^{x} \theta^{\alpha+(t+1)^\beta-t^\beta}\right\}_{x\geq 1}$ is an increasing sequence.

4.4.2 Properties of the DMW Type I Distribution

Substituting Equation (4.56) into Equation (4.10) yields the r^{th} moment about zero of the DMW (I) (α, β, θ) distribution in the form

$$\mu'_r = E(X^r) = \sum_{x=1}^{\infty}(x^r-(x-1)^r)\theta^{\alpha x+x^\beta}. \tag{4.65}$$

The first four moments about zero of the DMW (I) (α, β, θ) distribution follows directly from Equation (4.65) in the form

$$\left.\begin{aligned}
\mu_1' &= \sum_{x=1}^{\infty} \theta^{\alpha x + x^\beta} \\
\mu_2' &= \sum_{x=1}^{\infty} (2x-1)\, \theta^{\alpha x + x^\beta} \\
\mu_3' &= \sum_{x=1}^{\infty} (3x^2 - 3x + 1) \theta^{\alpha x + x^\beta} \\
\mu_4' &= \sum_{x=1}^{\infty} (4x^3 - 6x^2 + 4x - 1) \theta^{\alpha x + x^\beta}
\end{aligned}\right\} \tag{4.66}$$

Some moments, central moments, skewness, and kurtosis are evaluated at different values of the parameters.

The median of the DMW (I) (α, β, θ) distribution can be determined by solving the equation

$$\alpha(m+1) + (m+1)^\beta + \frac{\ln 2}{\ln \theta} = 0. \tag{4.67}$$

Proof:

$$P(X \leq m) \geq \frac{1}{2}$$

$$\xrightarrow{yields} 1 - \theta^{\alpha(m+1)+(m+1)^\beta} \geq \frac{1}{2}$$

$$\xrightarrow{yields} \theta^{\alpha(m+1)+(m+1)^\beta} \leq \frac{1}{2}$$

$$\xrightarrow{yields} \left(\alpha(m+1) + (m+1)^\beta\right) \ln \theta \leq \ln \frac{1}{2}$$

$$\xrightarrow{yields} \alpha(m+1) + (m+1)^\beta \leq \frac{\ln\left(\frac{1}{2}\right)}{\ln \theta}$$

$$\xrightarrow{yields} \alpha(m+1) + (m+1)^\beta + \frac{\ln 2}{\ln \theta} \leq 0.$$

Similarly, when $P(X \geq m) \leq \frac{1}{2}$ follows the relation

$$\alpha(m+1) + (m+1)^\beta + \frac{\ln 2}{\ln \theta} \geq 0.$$

The median of the DMW (I) (α, β, θ) distribution can be obtained by solving equation (4.67). For selected values of the parameters (α, β, θ) = (1.365, 1, 0.864) the median is m = 1.

The mode D of the DMW (I) (α, β, θ) distribution can be located graphically. The mode values corresponding to $(\alpha, \beta, \theta) =$ (1.136, 4, 0.803), (0.5, 1, 0.607), (0.25, 0.15, 0.803) are respectively, D = 1, 0, 0. This is illustrated in Figure (4.9).

Substituting Equation (4.56) into Equation (4.11) gives the moment generating function of the DMW (I) (α, β, θ) distribution in the form

$$M_X(t) = 1 + \sum_{x=1}^{\infty}\left(e^{xt} - e^{(x-1)t}\right)\theta^{\alpha x + x^{\beta}}. \tag{4.68}$$

Differentiating both sides of Equation (4.68) *r* times with respect to *t* gives

$$M_X^{(r)}(t) = \sum_{x=1}^{\infty}\left(x^r e^{xt} - (x-1)^r e^{(x-1)t}\right)\theta^{\alpha x + x^{\beta}}, \; r=1,2,.. \tag{4.69}$$

Substituting $t = 0$ and $r = 1, 2, 3, 4$, into Equation (4.69) gives the first four moments of the DMW (I) (α, β, θ) distribution in the form

$$\mu_1' = M_X'(t)_{|t=0} = \sum_{x=1}^{\infty} \theta^{x\alpha + x^{\beta}} \, ,$$

$$\mu_2' = M_X''(t)_{|t=0} = \sum_{x=1}^{\infty}(2x-1)\, \theta^{x\alpha + x^{\beta}} \, ,$$

$$\mu_3' = M_X^{(3)}(t)_{|t=0} = \sum_{x=1}^{\infty}(3x^2 - 3x + 1)\theta^{\alpha x + x^{\beta}},$$

$$\mu_4' = M_X^{(4)}(t)_{|t=0} = \sum_{x=1}^{\infty}(4x^3 - 6x^2 + 4x - 1)\theta^{\alpha x + x^{\beta}}.$$

When $\beta = 1$, the above calculated moments take the form

$$\mu_1' = M_X'(t)_{|t=0} = \sum_{x=1}^{\infty} \theta^{(\alpha+1)x} \, ,$$

$\mu_2' = M_X''(t)|_{t=0} = \sum_{x=1}^{\infty}(2x-1)\,\theta^{(\alpha+1)x}$,

$\mu_3' = M_X^{(3)}(t)|_{t=0} = \sum_{x=1}^{\infty}(3x^2-3x+1)\theta^{(\alpha+1)x}$,

$\mu_4' = M_X^{(4)}(t)|_{t=0} = \sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\theta^{(\alpha+1)x}$.

Substituting Equation (4.56) into Equation (4.12) gives the probability generating function of the DMW (I) (α,β,θ) distribution in the form

$$G_{[X]}(t) = 1 + (t-1)\sum_{x=1}^{\infty} t^{(x-1)}\,\theta^{\alpha x + x^{\beta}} \ . \qquad (4.70)$$

Calculating the first and second derivatives of the pgf $G_{[X]}(t)$ at t=1 gives the first and second factorial moments of the DMW (I) (α,β,θ) distribution in the form

$\mu_{[1]} = G'_{[X]}(1) = \sum_{x=1}^{\infty}\theta^{\alpha x + x^{\beta}}$,

$\mu_{[2]} = G''_{[X]}(1) = 2\sum_{x=1}^{\infty}(x-1)\,\theta^{\alpha x + x^{\beta}}$.

The mean, second moment, and variance of the DMW (I) (α,β,θ) distribution are respectively, given by

$E(X) = \mu_{[1]} = \sum_{x=1}^{\infty}\theta^{\alpha x + x^{\beta}}$,

$E(X^2) = \mu_{[1]} + \mu_{[2]} = \sum_{x=1}^{\infty}(2x-1)\,\theta^{x\alpha + x^{\beta}}$,

$Var(X) = \mu_{[1]} + \mu_{[2]} - \left(\mu_{[1]}\right)^2$

$$= \sum_{x=1}^{\infty}(2x-1)\,\theta^{x\alpha + x^{\beta}} - \left(\sum_{x=1}^{\infty}\theta^{x\alpha + x^{\beta}}\right)^2 . \qquad (4.71)$$

4.4.3 Estimation of the Parameters of the DMW (I) Distribution

The parameters α,β,θ of the DMW (I) distribution will be estimated by three methods of estimation.

(1) The Proportion Method

Case I: known parameters α, β and unknown parameter θ.

The proportion estimator of θ has the form

$$\theta^* = \left(1 - \frac{y}{n}\right)^{(\alpha+1)^{-1}}.$$

Proof: Let y be the number of the zero's in the sample

$$P(0; \alpha, \beta, \theta) = \left[\theta^{\alpha(0)+(0)^{\beta}} - \theta^{\alpha(0+1)+(0+1)^{\beta}}\right] = \frac{y}{n}$$

$$\xrightarrow{yields} [\theta^0 - \theta^{\alpha+1}] = \frac{y}{n} \tag{4.72}$$

$$\xrightarrow{yields} [1 - \theta^{\alpha+1}] = \frac{y}{n}$$

$$\xrightarrow{yields} \theta^{\alpha+1} = 1 - \frac{y}{n}$$

$$\xrightarrow{yields} (\alpha + 1) \ln \theta = \ln\left(1 - \frac{y}{n}\right)$$

$$\xrightarrow{yields} \ln \theta = \frac{\ln\left(1-\frac{y}{n}\right)}{(\alpha+1)}$$

$$\xrightarrow{yields} \theta^* = \left(1 - \frac{y}{n}\right)^{(\alpha+1)^{-1}}.$$

So, the unknown parameter θ has a proportion estimator θ^* in an exact solution when the two parameters α and β are known.

Case II: known parameter β and unknown parameters $\alpha \; and \; \theta$.

Let z be the number of the one's in the sample

$$\theta^{\alpha+1} - \theta^{2\alpha+2^{\beta}} = \frac{z}{n}. \tag{4.73}$$

Solving the Equations (4.72) and (4.73) gives the estimators $\alpha^*, \theta^*$of the parameters α, θ.

Case III: known parameter α and unknown parameters β *and* θ.

Solving the Equations (4.72), (4.73) gives the estimators β^*, θ^* of the parameters β, θ.

Case IV: known parameters $\alpha, \beta,$ *and* θ.

Let w be the number of the two's in the sample

$$\theta^{2\alpha+2^{\beta}} - \theta^{3\alpha+3^{\beta}} = \frac{w}{n}. \tag{4.74}$$

Solving the Equations (4.72)-(4.74) gives the proportion estimators $\alpha^*, \beta^*, \theta^*$.

(2) The Moments Method

Case I: known parameters α, β and unknown parameter θ.

Equating the first population moment to the first sample moment gives the equation

$$\sum_{x=0}^{\infty} x \left[\theta^{\alpha x + x^{\beta}} - \theta^{\alpha(x+1)+(x+1)^{\beta}}\right] = \frac{1}{n}\sum_{i=1}^{n} x_i . \tag{4.75}$$

Solving the Equation (4.75) gives the estimator $\tilde{\theta}$ of the parameter θ.

Case II: known parameter β and unknown parametersα *and* θ.

Equating the second population moment to the second sample moment gives

$$\sum_{x=0}^{\infty} x^2 \left[\theta^{\alpha x + x^{\beta}} - \theta^{\alpha(x+1)+(x+1)^{\beta}}\right] = \frac{1}{n}\sum_{i=1}^{n} x_i^2 . \tag{4.76}$$

Solving the Equations (4.75) and (4.76) numerically gives the estimators $\tilde{\alpha}, and\ \tilde{\theta}$ of the parameters $\alpha\ and \theta$.

Case III: known parameter α and unknown parameters $\beta\ and\ \theta$.

Solving the Equations (4.75) and (4.76) gives the moments estimators $\tilde{\beta}\ and\ \tilde{\theta}$ of the parameters $\beta\ and\ \theta$.

Case IV: known parameters $\alpha, \beta, and\ \theta$.

Equating the third population moment to the third sample moment gives the equation

$$\sum_{x=0}^{\infty} x^3 \left[\theta^{\alpha x+x^\beta} - \theta^{\alpha(x+1)+(x+1)^\beta}\right] = \frac{1}{n}\sum_{i=1}^{n} x_i^3 \ . \tag{4.77}$$

Solving the Equations (4.75)-(4.77) gives the moments estimators $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}$ of the parameters α, β, θ.

(3) The Maximum likelihood Method

The likelihood function and the log likelihood function of the DMW (I) (α, β, θ) distribution are

$$L = \prod_{i=1}^{n} \left[\theta^{\alpha x_i + x_i^\beta} - \theta^{\alpha(x_i+1)+(x_i+1)^\beta}\right],$$

$$\ln L = \sum_{i=1}^{n} \ln\left[\theta^{\alpha x_i + x_i^\beta} - \theta^{\alpha(x_i+1)+(x_i+1)^\beta}\right].$$

Case I: known parameters α, β and unknown parameter θ.

$$\frac{\partial\ lnL}{\partial \theta}\Big|_{\alpha=\hat{\alpha}, \beta=\hat{\beta}, \theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{(\alpha x_i + x_i{}^\beta)\theta^{\alpha x_i + x_i{}^\beta - 1} - [\alpha(x_i+1)+(x_i+1)^\beta]\theta^{\alpha(x_i+1)+(x_i+1)^\beta - 1}}{\left[\theta^{\alpha x_i + x_i{}^\beta} - \theta^{\alpha(x_i+1)+(x_i+1)^\beta}\right]} = 0\ . \tag{4.78}$$

Solving the Equation (4.78) gives the estimator $\hat{\theta}$ of the parameter θ.

Case II: known parameter β and unknown parameters α and θ.

$$\frac{\partial\, lnL}{\partial \alpha}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{\theta^{\alpha x_i + x_i^{\beta}}.x_i.\ln\theta - \theta^{\alpha(x_i+1)+(x_i+1)^{\beta}}.(x_i+1).\ln\theta}{\left[\theta^{\alpha x_i + x_i^{\beta}} - \theta^{\alpha(x_i+1)+(x_i+1)^{\beta}}\right]} = 0\ . \tag{4.79}$$

Solving the Equations (4.78) and (4.79) gives the estimator $\hat{\alpha}\ and\ \hat{\theta}$ of the parameters α and θ.

Case III: known parameter α and unknown parameters $\beta\ and\ \theta$.

$$\frac{\partial\, lnL}{\partial \beta}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{\theta^{\alpha x_i + x_i^{\beta}}(x_i)^{\beta}\ln(x_i) - \theta^{\alpha(x_i+1)+(x_i+1)^{\beta}}(x_i+1)^{\beta}\ln(x_i+1)}{\left[\theta^{\alpha x_i + x_i^{\beta}} - \theta^{\alpha(x_i+1)+(x_i+1)^{\beta}}\right]} = 0. \tag{4.80}$$

Solving the Equations (4.78), (4.80) gives the estimator $\hat{\beta}\ and\ \hat{\theta}$ of the parameters $\beta\ and\ \theta$.

Case IV: known parameters $\alpha, \beta, and\ \theta$.

Solving the Equations (4.78)-(4.80) gives the estimator $\hat{\alpha}, \hat{\beta}, and\ \hat{\theta}$ of the parameters $\alpha, \beta,$ and θ.

4.4.4 Special Distributions from DMW (I) Distribution

Many discretized distributions follows as special cases of DMW (I) (α, β, θ) distribution. Examples are the discretized linear failure rate, discretized Weibull, discretized Rayleigh, and discretized exponential distributions.

As one of our new results the discretized linear failure rate distribution is handled in detail. The remaining discretized distributions are referred shortly. This distribution follows from the DMW (I) (α, β, θ) distribution as special case when $\beta = 2$. The discrete linear failure rate distribution is referred to as DLFR (α, θ) distribution. This distribution corresponds to the case when

$$g(x) = \alpha x + x^2. \tag{4.81}$$

The pmf of the DLFR (α, θ) distribution has the form

$$P(x) = \theta^{\alpha x + x^2} - \theta^{\alpha(x+1)+(x+1)^2}, x = 0,1,2, \ldots \tag{4.82}$$

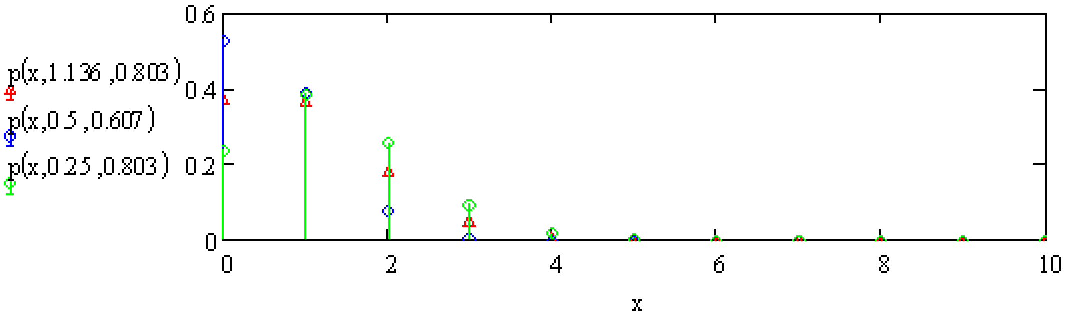


Figure 4.13 The pmf of DLFR distribution at different parameters' values.

As a special distribution of DMW (I) (α, β, θ) distribution, all the characteristics of the DLFR(α, θ) distribution are simply obtained from the characteristics of the DMW (I) (α, β, θ) distribution by substituting the value $\beta = 2$. When $\beta = 2$, Equation (4.59) gives the cumulative distribution function of the DLFR (α, θ) distribution

$$F_1(x) = 1 - \theta^{\alpha(x+1)+(x+1)^2}. \tag{4.83}$$

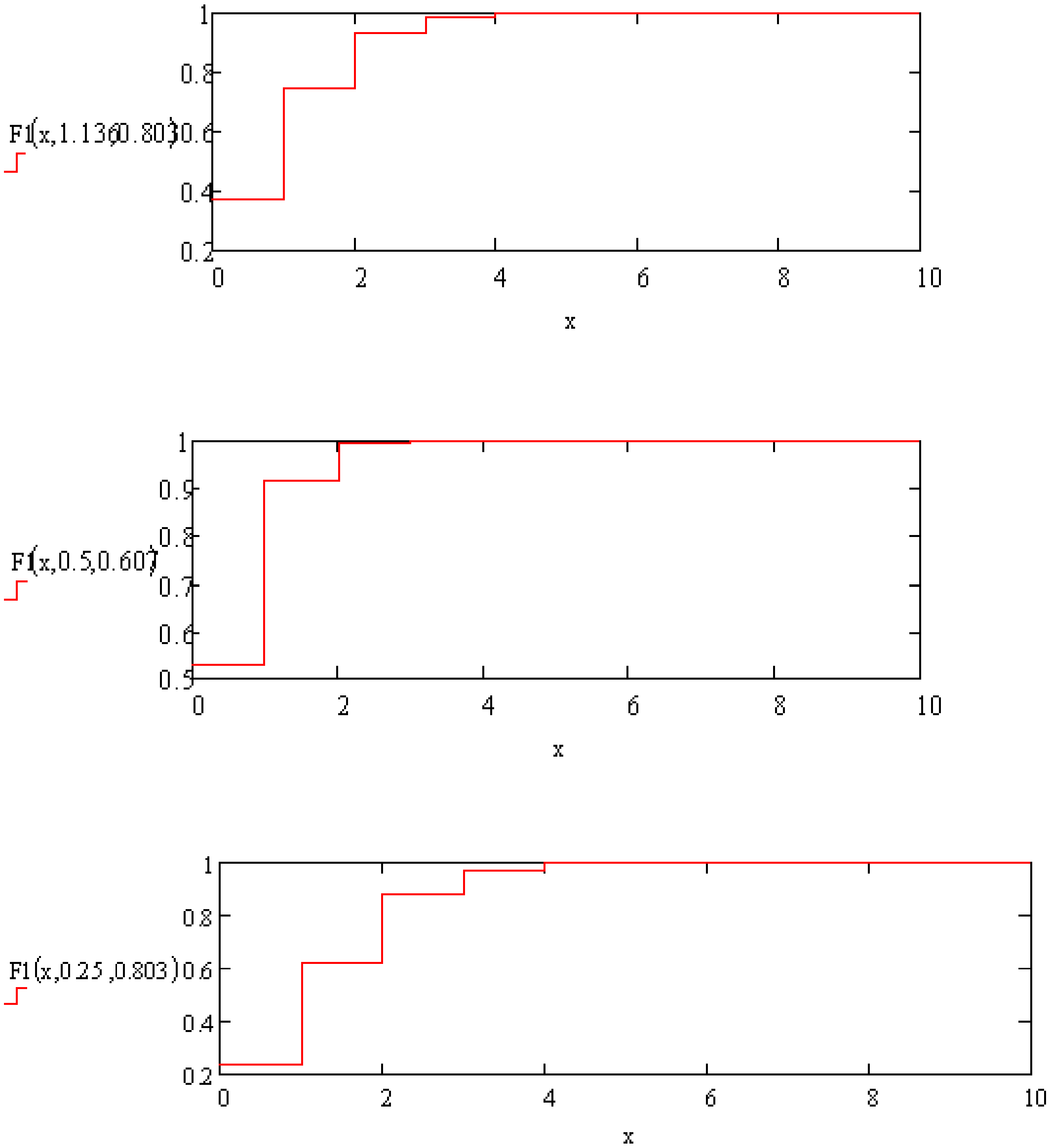


Figure 4.14 The cdf of DLFR distributions at different parameters' values.

The survival function of the DLFR (α, θ) distribution follows from Equation (4.60) when $\beta = 2$ in the form

$$S_1(x) = \theta^{\alpha(x+1)+(x+1)^2}. \tag{4.84}$$

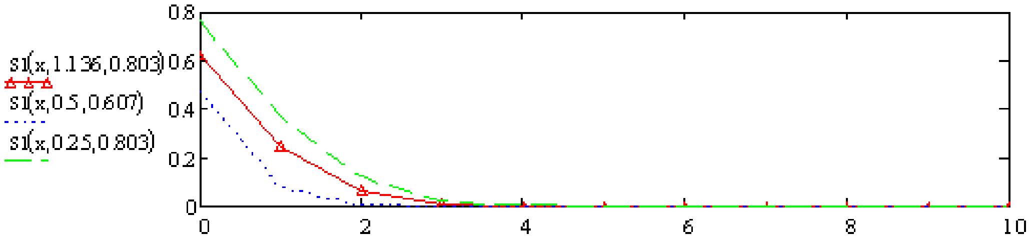


Figure 4.15 The survival function of DLFR distributions at different parameters' values

Substituting the value $\beta = 2$ into Equation (4.61) gives the failure rate function of the DLFR (α, θ) distribution in the form

$$r_1(x) = 1 - \theta^{\alpha+2x+1}. \tag{4.85}$$

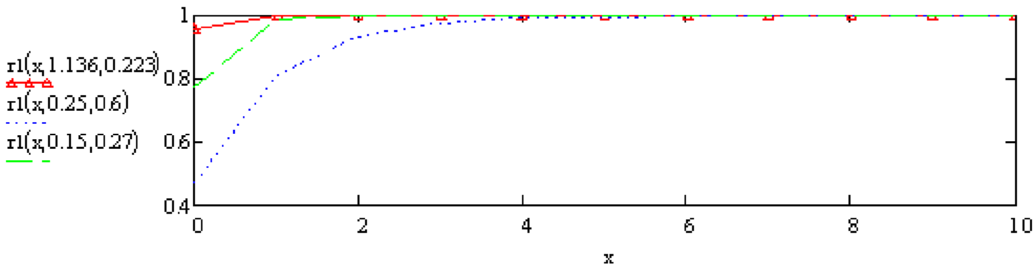


Figure 4.16 The failure rate function of DLFR distribution at different parameters' values

The second failure rate function of the DLFR (α, θ) distribution follows from Equation (4.62) when $\beta = 2$ in the form

$$SRF_1 = -[\alpha + 2x + 1] \ln \theta. \tag{4.86}$$

In light of Equation (4.63) when $\beta = 2$, the residual reliability function of the DLFR (α, θ) distribution is

$$R_1(i|x) = \theta^{\alpha i + 2i(x+1) + i^2}. \tag{4.87}$$

The cumulative hazard function of the DLFR(α, θ) distribution is

$$H_1(x) = x - \sum_{t=1}^{x} \theta^{\alpha+2t+1}. \tag{4.88}$$

The DLFR (α, θ) distribution has an increasing failure rate (IFR), since the following equivalent conditions are satisfied

IFR1: $\{1-\theta^{\alpha+2x+1}\}_{x\geq 1}$ is an increasing sequence.

IFR2: For all $i \in N^*\left\{\theta^{i\alpha+2i(x+1)+i^2}\right\}_{x\geq 1}$ is a decreasing sequence.

Also, the DLFR $(\alpha, \theta\ )$ distribution has an increasing failure rate in average (IFRA), since the following equivalent conditions are satisfied

IFRA1: $\left\{\left(\theta^{\alpha(x+1)+(x+1)^2}\right)^{1/x}\right\}_{x\geq 1}$ is a decreasing sequence.

IFRA2: $\left\{1-\frac{1}{x}\sum_{t=1}^{x}\theta^{\alpha+2t+1}\right\}_{x\geq 1}$ is an increasing sequence.

The r^{th} moments about zero of the DLFR (α, θ) distribution follows from Equation (4.65), when $\beta = 2$ in the form

$$\mu_r' = E(X^r) = \sum_{x=1}^{\infty}(x^r-(x-1)^r)\theta^{\alpha x+x^2}. \tag{4.89}$$

From Equation (4.89) the moments μ_r' can be easily obtained by substituting the suitable value of *r* into this equation.

$$\mu_1' = \sum_{x=1}^{\infty}\theta^{\alpha x+x^2},$$
$$\mu_2' = \sum_{x=1}^{\infty}(2x-1)\,\theta^{\alpha x+x^2}\ ,$$
$$\mu_3' = \sum_{x=1}^{\infty}(3x^2-3x+1)\theta^{\alpha x+x^2},$$
$$\mu_4' = \sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\theta^{\alpha x+x^2}\ .$$

Some moments, central moments, skewness, and kurtosis are evaluated at different parameters values.

The median of the DLFR (α, θ) distribution is determined, at selected values of the parameters α, θ by the numerical solution of the equation

$\alpha(m+1)+(m+1)^2+\frac{\ln 2}{\ln\theta}=0.$

Or, equivalently

$$m^2+(\alpha+2)m+\left(1+\alpha+\frac{\ln 2}{\ln\theta}\right)=0. \qquad (4.90)$$

The median m of the DLFR (I) (α,θ) distribution can be obtained by solving Equation (4.90). For selected values of the parameters (α,θ) = (8.849, 0.986) m = 6. The mode of the DLFR (α,θ) distribution can be located graphically. The mode values corresponding to (α,θ) = (1.136, 0.803), (0.5, 0.607), (0.25, 0.803) are respectively, D = 0, 0, 1. This is illustrated in Figure (4.13).

The moment generating function of the DLFR (α,θ) distribution has the form

$$M_X(t)=E(e^{tX})=1+\sum_{x=1}^{\infty}\left(e^{xt}-e^{(x-1)t}\right)\theta^{\alpha x+x^2}. \qquad (4.91)$$

Differentiating Equation (4.91), gives the r^{th} derivative of the mgf in the form

$$M_X^{(r)}(t)=\sum_{x=1}^{\infty}\left(x^r e^{xt}-(x-1)^r e^{(x-1)t}\right)\theta^{\alpha x+x^2},\ r=1,2,\ldots \qquad (4.92)$$

The first four moments μ_r', r = 1, 2, 3, 4 can be easily obtained by simply substituting these values of r into Equation (4.92).

$$\mu_1'=M_X'(t)|_{t=0}=\sum_{x=1}^{\infty}\theta^{x\alpha+x^2},$$

$$\mu_2'=M_X''(t)|_{t=0}=\sum_{x=1}^{\infty}(2x-1)\,\theta^{x\alpha+x^2},$$

$$\mu_3'=M_X^{(3)}(t)|_{t=0}=\sum_{x=1}^{\infty}(3x^2-3x+1)\theta^{\alpha x+x^2},$$

$$\mu_4'=M_X^{(4)}(t)|_{t=0}=\sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\theta^{\alpha x+x^2}.$$

The probability generating function of the DLFR (α,θ) distribution is easily obtained, by substituting $\beta = 2$ into Equation (4.70), in the form

$$G_{[X]}(t) = E(t^X) = 1 + (t-1)\sum_{x=1}^{\infty} t^{(x-1)}\,\theta^{\alpha x + x^2}. \tag{4.93}$$

From Equation (4.93) the mean, second moment, and the variance can be obtained, in the form

$$E(X) = \sum_{x=1}^{\infty} \theta^{x\alpha + x^2},$$

$$E(X^2) = \sum_{x=1}^{\infty}(2x-1)\,\theta^{x\alpha + x^2},$$

$$Var(X) = \sum_{x=1}^{\infty}(2x-1)\,\theta^{x\alpha + x^2} - \left(\sum_{x=1}^{\infty} \theta^{x\alpha + x^2}\right)^2. \tag{4.94}$$

The parameters of the DLFR(α,θ) distributions are estimated by the proportion, the moments, and the maximum likelihood methods.

(1) The Proportion Method

Case I: known parameter α and unknown parameter θ.

This case of DMW (I) (α,β,θ) when α, $\beta = 2$ are known and θ is unknown. Therefore, the unknown parameter θ has a proportion estimator θ^* of the form

$$\theta^* = \left(1 - \frac{y}{n}\right)^{(1+\alpha)^{-1}}.$$

Proof: Let y be the number of the zero's in the sample

$$P(0;\theta,\alpha) = \left[\theta^{\alpha(0)+(0)^2} - \theta^{\alpha(0+1)+(0+1)^2}\right] = \frac{y}{n}$$

$$\xrightarrow{yields} [\theta^0 - \theta^{\alpha+1}] = \frac{y}{n}$$

$$\xrightarrow{yields} [1 - \theta^{\alpha+1}] = \frac{y}{n} \tag{4.95}$$

$$\xrightarrow{yields} \theta^{\alpha+1} = 1 - \frac{y}{n} \tag{4.96}$$

$$\xrightarrow{yields} (\alpha + 1) \ln \theta = \ln\left(1 - \frac{y}{n}\right)$$

$$\xrightarrow{yields} \ln \theta = \frac{\ln\left(1-\frac{y}{n}\right)}{(\alpha+1)}$$

$$\xrightarrow{yields} \theta^* = e^{\left(\frac{\ln\left(1-\frac{y}{n}\right)}{(\alpha+1)}\right)}$$

$$\xrightarrow{yields} \theta^* = \left(1 - \frac{y}{n}\right)^{(1+\alpha)^{-1}}.$$

Case II: unknown parameters α and θ.

Let z be the number of the one's in the sample, i,e.

$$P(1;\theta,\alpha) = [\theta^{\alpha+1} - \theta^{2\alpha+4}] = \frac{z}{n}. \tag{4.97}$$

The proportion estimator for α , θ are given by

$$\theta^* = \frac{\sqrt{\left(1-\frac{y}{n}\right)-\frac{z}{n}}}{\left(1-\frac{y}{n}\right)}, \alpha^* = \frac{\ln\left(1-\frac{Y}{n}\right)}{\ln \theta^*} - 1 .$$

Proof:

From (4.96) in (4.97)

$$\xrightarrow{yields} \left[\theta^{\alpha+1} - \theta^{2(\alpha+1)+2}\right] = \frac{z}{n}$$

$$\xrightarrow{yields} \left(1 - \frac{Y}{n}\right) - \left(1 - \frac{Y}{n}\right)^2 \theta^2 = \frac{Z}{n}$$

$$\xrightarrow{yields} \left(1 - \frac{Y}{n}\right)^2 \theta^2 = \left(1 - \frac{Y}{n}\right) - \frac{Z}{n}$$

$$\xrightarrow{yields} \theta^2 = \frac{\left(1-\frac{Y}{n}\right)-\frac{Z}{n}}{\left(1-\frac{Y}{n}\right)^2}$$

$$\xrightarrow{yields} \theta^* = \frac{\sqrt{\left(1-\frac{Y}{n}\right)-\frac{Z}{n}}}{\left(1-\frac{Y}{n}\right)}. \tag{4.98}$$

From (4.98) in (4.96)

$$\xrightarrow{yields} (\alpha+1)\ln\theta^* = \ln\left(1-\frac{Y}{n}\right)$$

$$\xrightarrow{yields} (\alpha+1) = \frac{\ln\left(1-\frac{Y}{n}\right)}{\ln\theta^*}$$

$$\xrightarrow{yields} \alpha^* = \frac{\ln\left(1-\frac{Y}{n}\right)}{\ln\theta^*} - 1.$$

(2) The Moments Method

Case I: known parameter α and unknown parameter θ.

Equating the first population moment to the first sample moment gives the equation

$$\sum_{i=1}^{\infty}\theta^{i\alpha+i^2} = \frac{1}{n}\sum_{i=0}^{n} x_i \ . \qquad (4.99)$$

Solving the Equation (4.99) gives the estimator $\tilde{\theta}$ of the parameter θ.

Case II: unknown parameters $\alpha \ and \ \theta$.

Equating the second population moment to the second sample moment gives

$$\sum_{i=1}^{\infty}(2i-1)\theta^{i\alpha+i^2} = \frac{1}{n}\sum_{i=0}^{n} x_i^2 \ . \qquad (4.100)$$

Solving the Equations (4.99) and (4.100) gives the estimator $\tilde{\alpha}, \ and \ \tilde{\theta}$ of the parameters $\alpha \ and \ \theta$.

(3) The Maximum likelihood Method

The likelihood and the log likelihood functions of the DLFR (α, θ) distribution are

$$L = \prod_{i=1}^{n}\left[\theta^{\alpha x_i + x_i^2} - \theta^{\alpha(x_i+1)+(x_i+1)^2}\right],$$

$$\ln L = \sum_{i=1}^{n}\ln\left[\theta^{\alpha x_i + x_i^2} - \theta^{\alpha(x_i+1)+(x_i+1)^2}\right].$$

Case I: known parameter α and unknown parameter θ.

$$\frac{\partial \, lnL}{\partial \theta}|_{\alpha=\hat{\alpha},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{(\alpha x_i + x_i^2)\theta^{\alpha x_i + x_i^2 - 1} - \alpha(x_i+1) + (x_i+1)^2 \theta^{\alpha(x_i+1)+(x_i+1)^2 - 1}}{\left[\theta^{\alpha x_i + x_i^2} - \theta^{\alpha(x_i+1)+(x_i+1)^2}\right]} = 0. \tag{4.101}$$

Solving the Equation (4.101) gives the estimator $\hat{\theta}$ of the parameter θ.

Case II: unknown parameters α, θ.

$$\frac{\partial \, lnL}{\partial \alpha}|_{\alpha=\hat{\alpha},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{\theta^{\alpha x_i + x_i^2} . x_i - \theta^{\alpha(x_i+1)+(x_i+1)^2} . (x_i+1)}{\left[\theta^{\alpha x_i + x_i^2} - \theta^{\alpha(x_i+1)+(x_i+1)^2}\right]} = 0. \tag{4.102}$$

Solving the Equations (4.101) and (4.102) gives the estimators $\hat{\alpha} \; and \; \hat{\theta}$ of the parameters $\alpha \; and \; \theta$.

As a second special case of the DMW (I) (α, β, θ) distribution, we briefly refer to the discrete Weibull distribution. This distribution is denoted by DW (β, θ) distribution. This distribution is obtained from DMW (I) (α, β, θ) distribution when $\alpha = 0$.

So, the cdf becomes

$F_1(x) = 1 - \theta^{(x+1)^\beta}, \theta = e^{-\lambda}$.

(Nakagawa and Osaki (1975))

As a third special case of the DMW (I) (α, β, θ) distribution, we briefly refer to the discrete Rayleigh distribution. This distribution is denoted by DR (θ) distribution. The DR (θ) distribution is obtained from DMW (I) (α, β, θ) distribution when $\alpha = 0$ and $\beta = 2$. The cdf is

$F_1(x) = 1 - \theta^{(x+1)^2}, \theta = e^{-\lambda}.$

(Roy (2004))

As a fourth special case of the DMW (I) (α, β, θ) distribution, we briefly refer to the discrete exponential distribution. This distribution is denoted by DE (θ) distribution. The DE (θ) distribution is obtained from DMW (I) (α, β, θ) distribution when $\beta = 0$. The cdf has the form

$F_1(x) = 1 - \theta^{(x+1)}, \theta = e^{-\alpha}.$

4.5 Discrete Modified Weibull Type II Distribution

Using the first discretizing approach, introduced in Section 3.2 of Chapter III, of the general class of continuous distributions and putting

$$g(x) = x^{\beta} e^{\alpha x}, \qquad (4.103)$$

we obtain the survival function of the MW (II) (α, β, λ) distribution as follows

$S(x) = e^{-\lambda x^{\beta} e^{\alpha x}}.$

Using the notation $\theta = e^{-\lambda}, (0 < \theta < 1)$, the survival function is

$S(x) = \theta^{x^{\beta} e^{\alpha x}}.$

By using Equation (4.3), the probability mass function of the DMW (II) (α, β, θ) distribution is

$$P(x) = \theta^{x^{\beta} e^{\alpha x}} - \theta^{(x+1)^{\beta} e^{\alpha(x+1)}}, x = 0,1,2, \ldots \qquad (4.104)$$

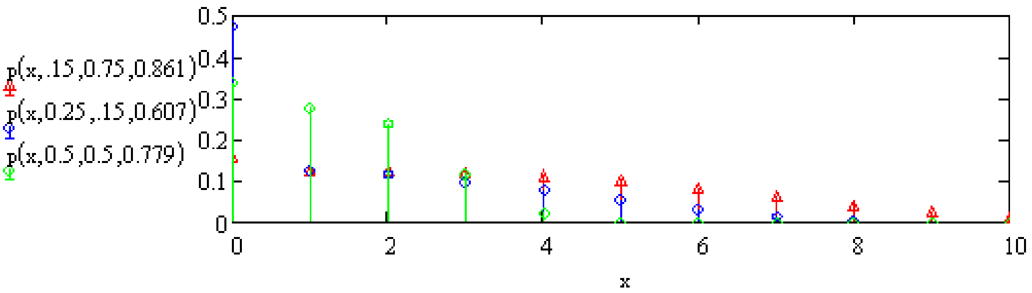


Figure 4.17 The pmf of DMW(II) distribution at different parametes' values.

To prove that $P(x)$ is a probability mass function it should be

(i) $P(x) \geq 0, trivial, since\ 0 < \theta < 1, \beta > 0, \alpha > 0$

(ii) $\sum_{x=0}^{\infty} P(x) = 1,$

Proof:

$$\sum_{x=0}^{\infty} P(x) = \sum_{x=0}^{\infty} \left[\theta^{x^{\beta} e^{\alpha x}} - \theta^{(x+1)^{\beta} e^{\alpha(x+1)}}\right]$$

$$= \left(\theta^{0^{\beta} e^{\alpha(0)}} + \theta^{1^{\beta} e^{\alpha(1)}} + \theta^{2^{\beta} e^{\alpha(2)}} + \cdots\right)$$

$$-\left(\theta^{1^{\beta} e^{\alpha(1)}} + \theta^{2^{\beta} e^{\alpha(2)}} + \theta^{3^{\beta} e^{\alpha(3)}} + \cdots\right)$$

$$= \theta^{0} = e^{-\alpha(0)} = 1.$$

Substituting Equation (4.103) into Equation (4.4) gives the cumulative distribution function of the DMW (II) (α, β, θ) distribution in the form

$$F_1(x) = 1 - \theta^{(x+1)^{\beta} e^{\alpha(x+1)}}. \qquad (4.105)$$

This is the cdf of MW (II) (α, β, λ) distribution calculated at $(x + 1)$. It is clear that

$\lim_{x \to 0^{+}} F_1(x) = 0$ and $\lim_{x \to \infty^{-}} F_1(x) = 1.$

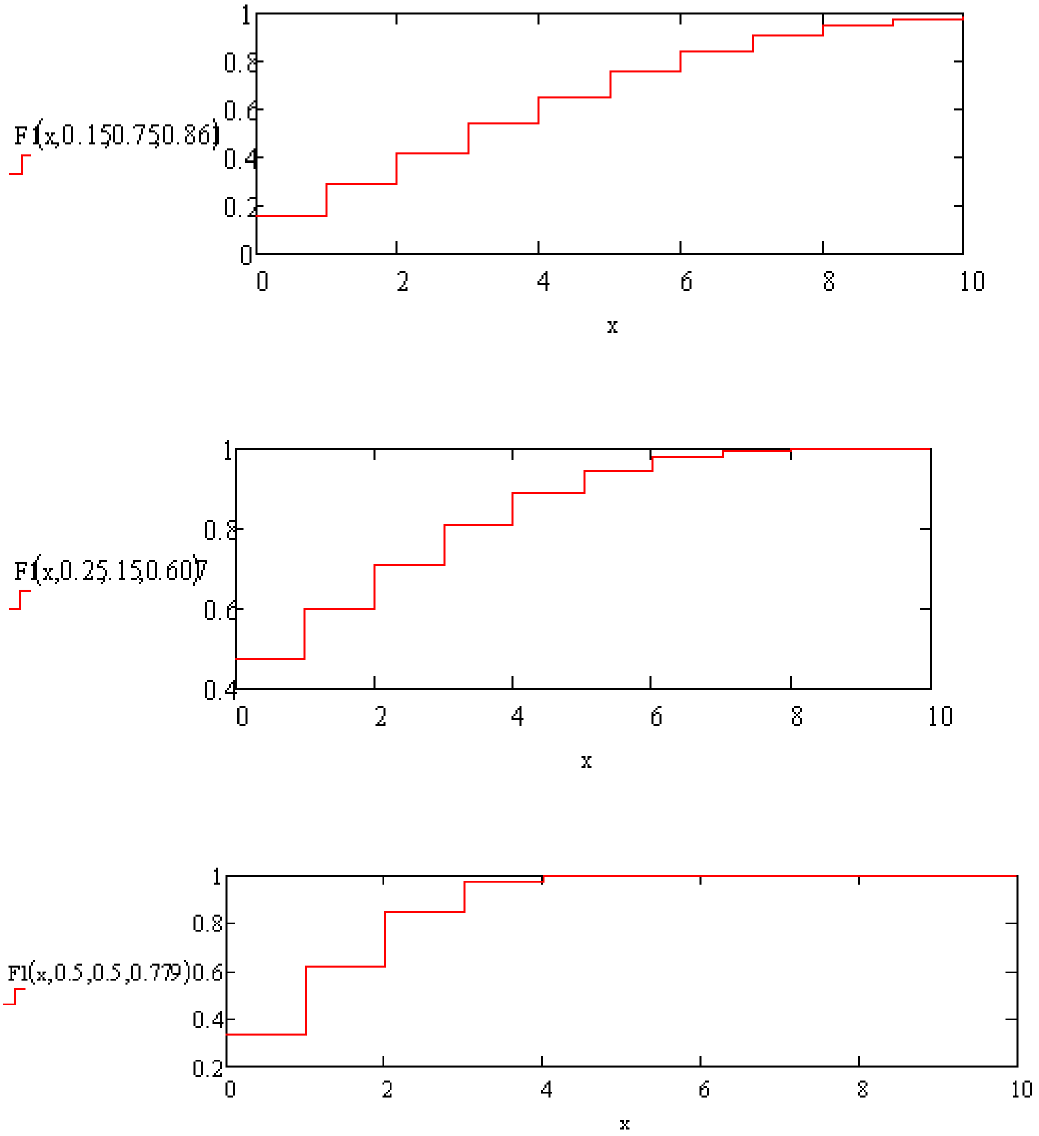


Figure 4.18 The cdf of DMW(II) distributions at different parameterars' values.

4.5.1 Some Reliability Measures of the DMW (II) Distribution

Substituting Equation (4.103) into Equation (4.5) gives the survival function of the DMW (II) (α, β, θ) distribution in the form

$$S_1(x) = \theta^{(x+1)^{\beta} e^{\alpha(x+1)}}. \tag{4.106}$$

This is the same as the survival function in the continuous case at $(x + 1)$.

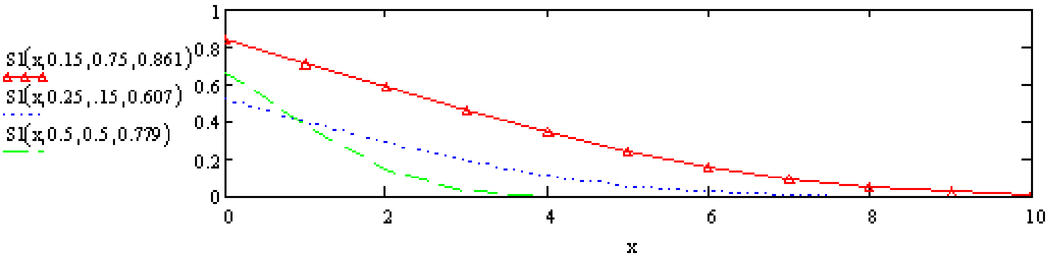


Figure 4.19 The survival function of DMW(II) distributions at different parameters' values

Substituting Equation (4.103) into Equation (4.6) gives the failure rate function of the DMW (II) (α, β, θ) distribution in the form

$$r_1(x) = 1 - \theta^{(x+1)^\beta e^{\alpha(x+1)} - x^\beta e^{\alpha x}}. \tag{4.107}$$

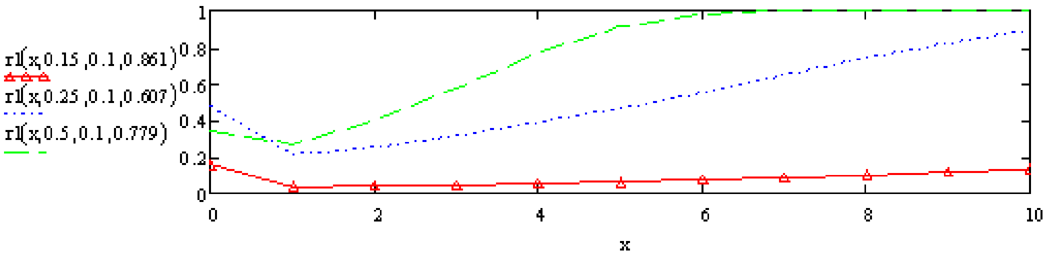


Figure 4.20 The failure rate function of DMW(II) distribution at different parameters' values

Substituting Equation (4.103) into Equation (4.7) gives the second failure rate function of the DMW (II) (α, β, θ) distribution in the form

$$SRF_1(x) = \left(x^\beta e^{\alpha x} - (x+1)^\beta e^{\alpha(x+1)}\right) \ln \theta. \tag{4.108}$$

Substituting Equation (4.103) into Equation (4.8) gives the residual reliability function of the DMW (II) (α, β, θ) distribution in the form

$$R_1(i|x) = \theta^{(x+i+1)^\beta e^{\alpha(x+i+1)} - (x+1)^\beta e^{\alpha(x+1)}}. \tag{4.109}$$

Substituting Equation (4.103) into Equation (4.9) gives the cumulative hazard function of the DMW (II) (α, β, θ) distribution in the form

$$H_1(x) = x - \sum_{t=1}^{x} \theta^{(t+1)^\beta e^{\alpha(t+1)} - t^\beta e^{\alpha t}}. \tag{4.110}$$

The DMW (II) (α, β, θ) distribution has an increasing failure rate (IFR), since the following equivalent conditions are satisfied

IFR1: $\left\{1 - \theta^{(x+1)^{\beta} e^{\alpha(x+1)} - x^{\beta} e^{\alpha x}}\right\}_{x \geq 1}$ is an increasing sequence.

IFR2: For all $i \in N^{*}$ $\left\{\theta^{(x+i+1)^{\beta} e^{\alpha(x+i+1)} - (x+1)^{\beta} e^{\alpha(x+1)}}\right\}_{x \geq 1}$ is a decreasing sequence.

Also, the DMW (II) (α, β, θ) distribution has an increasing failure rate in average (IFRA), since the following equivalent conditions are satisfied

IFRA1: $\left\{\left(\theta^{(x+1)^{\beta} e^{\alpha(x+1)}}\right)^{1/x}\right\}_{x \geq 1}$ is a decreasing sequence.

IFRA2: $\left\{1 - \frac{1}{x} \sum_{t=1}^{x} \theta^{(t+1)^{\beta} e^{\alpha(t+1)} - t^{\beta} e^{\alpha t}}\right\}_{x \geq 1}$ is an increasing sequence.

4.5.2 Properties of the DMW (II) Distribution

Substituting Equation (4.103) into Equation (4.10) gives the r^{th} moments about zero of the DMW (II) (α, β, θ) distribution in the form

$$\mu_r' = E(X^r) = \sum_{x=1}^{\infty} (x^r - (x-1)^r) \theta^{x^{\beta} e^{\alpha x}}. \quad (4.111)$$

Equation (4.111) gives the first four moments about zero of the DMW (II) (α, β, θ) distribution by substituting the value r = 1, 2, 3, 4.

$$\mu_1' = E(X) = \sum_{x=1}^{\infty} \theta^{x^{\beta} e^{\alpha(x)}},$$

$$\mu_2' = E(X^2) = \sum_{x=1}^{\infty} (2x - 1) \theta^{x^{\beta} e^{\alpha(x)}},$$

$$\mu_3' = E(X^3) = \sum_{x=1}^{\infty} (3x^2 - 3x + 1) \theta^{x^{\beta} e^{\alpha x}},$$

$$\mu_4' = E(X^4) = \sum_{x=1}^{\infty} (4x^3 - 6x^2 + 4x - 1) \theta^{x^{\beta} e^{\alpha x}}.$$

The median of the DMW (II) (α,β,θ) distribution can be determined by solving the equation

$$(m+1)^{\beta}e^{\alpha(m+1)}+\frac{\ln(2)}{\ln\theta}=0. \qquad (4.112)$$

Proof:

$$P(X\le m)\ge\frac{1}{2}\xrightarrow{yields}1-\theta^{(m+1)^{\beta}e^{\alpha(m+1)}}\ge\frac{1}{2}$$

$$\xrightarrow{yields}-\theta^{(m+1)^{\beta}e^{\alpha(m+1)}}\ge-\frac{1}{2}$$

$$\xrightarrow{yields}\theta^{(m+1)^{\beta}e^{\alpha(m+1)}}\le\frac{1}{2}$$

$$\xrightarrow{yields}(m+1)^{\beta}e^{\alpha(m+1)}\ln\theta\le\ln\left(\frac{1}{2}\right)$$

$$\xrightarrow{yields}(m+1)^{\beta}e^{\alpha(m+1)}\le\frac{\ln\left(\frac{1}{2}\right)}{\ln\theta}$$

The median of the DMW (II) (α,β,θ) distribution can be obtained by solving equation (4.112). For selected values of the parameters (α,β,θ) = (0.01, 1, 0.897) the median is m = 5.

The mode of the DMW (II) (α,β,θ) distribution can be located graphically. The mode values corresponding to (α,β,θ) = (0.15, 0.75, 0.861), (0.25, 0.15, 0.607), (0.5, 0.5, 0.779), the mode values are D = 0, 0, 0. This is illustrated in Figure (4.17).

Substituting Equation (4.103) into Equation (4.11) gives the moments generating function of the DMW (II) (α,β,θ) distribution in the form

$$M_X(t)=1+\sum_{x=1}^{\infty}\left(e^{xt}-e^{(x-1)t}\right)\theta^{x^{\beta}e^{\alpha x}}. \qquad (4.113)$$

Calculating the derivatives of the moment generating function at $t = 0$ gives the first four moments of the DMW (II) (α, β, θ) distribution in the form

$$\mu_1' = M_X'(t)_{|t=0} = \sum_{x=1}^{\infty} \theta^{x^\beta e^{\alpha(x)}},$$

$$\mu_2' = M_X''(t)_{|t=0} = \sum_{x=1}^{\infty}(2x-1)\theta^{x^\beta e^{\alpha(x)}},$$

$$\mu_3' = M_X^{(3)}(t)_{|t=0} = \sum_{x=1}^{\infty}(3x^2-3x+1)\theta^{x^\beta e^{\alpha(x)}},$$

$$\mu_4' = M_X^{(4)}(t)_{|t=0} = \sum_{x=1}^{\infty}(4x^3-6x^2+4x-1)\theta^{x^\beta e^{\alpha(x)}}.$$

Substituting Equation (4.103) into Equation (4.12) yields the probability generating function of the DMW (II) (α, β, θ) distribution in the form

$$G_{[X]}(t) = 1 + (t-1)\sum_{x=1}^{\infty} t^{(x-1)}\, \theta^{x^\beta e^{\alpha x}} \,. \qquad (4.114)$$

Calculating the first and second derivatives at t=1 gives the first and the second factorial moments of the DMW (II) (α, β, θ) distribution in the form

$$\mu_{[1]} = G'_{[X]}(t) = \sum_{x=1}^{\infty} \theta^{x^\beta e^{\alpha(x)}},$$

$$\mu_{[2]} = G''_{[X]}(t) = 2\sum_{x=1}^{\infty}(x-1)\, \theta^{x^\beta e^{\alpha(x)}}.$$

The second moment and variance of the DMW (II) (α, β, θ) distribution are

$$E(X^2) = \sum_{x=1}^{\infty}(2x-1)\, \theta^{x^\beta e^{\alpha(x)}},$$

$$Var(X) = \sum_{x=1}^{\infty}(2x-1)\, \theta^{x^\beta e^{\alpha(x)}} - \left(\sum_{x=1}^{\infty} \theta^{x^\beta e^{\alpha(x)}}\right)^2.$$

4.5.3 Estimation of the Parameters of the DMW (II) Distribution

The parameters α, β, θ of the DMW (II) distribution will be estimated by:

(1) The Proportion Method

Case I: known parameters $\alpha \; and \; \beta$ and unknown parameter θ.

The proportion estimator of θ has the form

$\theta^* = \left(1 - \frac{y}{n}\right)^{e^{-\alpha}}.$

Proof: Let y be the number of the zero's in the sample

$$P(0;\alpha,\beta,\theta) = \left[\theta^{0^\beta e^{\alpha(0)}} - \theta^{1^\beta e^{\alpha(1)}}\right] = \frac{y}{n}$$

$$\xrightarrow{yields} \left[\theta^0 - \theta^{e^\alpha}\right] = \frac{y}{n} \qquad (4.115)$$

$$\xrightarrow{yields} \left[1 - \theta^{e^\alpha}\right] = \frac{y}{n}$$

$$\xrightarrow{yields} \theta^{e^\alpha} = 1 - \frac{y}{n}$$

$$\xrightarrow{yields} e^\alpha \ln\theta = \ln(1 - \frac{y}{n})$$

$$\xrightarrow{yields} \ln\theta = \frac{\ln(1-\frac{y}{n})}{e^\alpha}$$

$$\xrightarrow{yields} \theta^* = e^{\left(\frac{\ln(1-\frac{y}{n})}{e^\alpha}\right)}$$

$$\xrightarrow{yields} \theta^* = \left(1 - \frac{y}{n}\right)^{e^{-\alpha}}.$$

So, the unknown parameter has a proportion estimator in an exact form when the two parameters α and β are known.

Case II: known parameter β and unknown parameters α and θ.

Let z be the number of the one's in the sample

$$P(1;\alpha,\beta,\theta) = \left[\theta^{1^\beta e^{\alpha(1)}} - \theta^{2^\beta e^{\alpha(2)}}\right] = \frac{z}{n}$$

$$\xrightarrow{yields} \left[\theta^{e^\alpha} - \theta^{2^\beta e^{2\alpha}}\right] = \frac{z}{n}. \qquad (4.116)$$

Solving the Equations (4.115) and (4.116) numerically gives the estimators $\alpha^*\ and\ \theta^*$ of the parameters α and θ.

Case III: known parameter α and unknown parameters $\beta \ and \ \theta$.

Solving the Equations (4.115) and (4.116) gives the estimators $\beta^{*} and \ \theta^{*}$ of the parameters $\beta \ and \ \theta$.

Case IV: unknown parameters $\alpha, \beta, and \ \theta$.

Let w be the number of the two's in the sample

$$P(2;\alpha,\beta,\theta)=\left[\theta^{2^{\beta}e^{\alpha(2)}}-\theta^{3^{\beta}e^{\alpha(3)}}\right]=\frac{w}{n}$$

$$\xrightarrow{yields}\left[\theta^{2^{\beta}e^{2\alpha}}-\theta^{3^{\beta}e^{3\alpha}}\right]=\frac{w}{n}. \qquad (4.117)$$

Solving the Equations (4.115) - (4.117) numerically yields the estimators $\alpha^{*}, \beta^{*}, and \ \theta^{*}$ of the parameters $\alpha, \beta, and \ \theta$.

(2) The Moments Method

Case I: known parameters $\alpha \ and \ \beta$ and unknown parameter θ.

Equating the first population moment to the first sample moment gives the equation

$$\sum_{x=0}^{\infty} x\left[\theta^{(x)^{\beta}e^{\alpha(x)}}-\theta^{(x+1)^{\beta}e^{\alpha(x+1)}}\right]=\frac{1}{n}\sum_{i=1}^{n} x_i \ . \qquad (4.118)$$

Solving the Equation (4.118) gives the estimator $\tilde{\theta}$ of the parameter θ.

Case II: known parameter β and unknown parameters α andθ.

Equating the second population moment to the second sample moment gives

$$\sum_{x=0}^{\infty} x^2\left[\theta^{(x)^{\beta}e^{\alpha(x)}}-\theta^{(x+1)^{\beta}e^{\alpha(x+1)}}\right]=\frac{1}{n}\sum_{i=1}^{n} x_i^2 \ . \qquad (4.119)$$

Solving the Equations (4.118) and (4.119) gives the estimators $\tilde{\alpha} \ and \ \tilde{\theta}$ of the parameters α and θ.

Case III: known parameter α and unknown parameters $\beta\ and\ \theta$.

Solving the Equations (4.118) and (4.119) gives the estimators $\tilde{\beta}\ and\ \tilde{\theta}$ of the parameters $\beta\ and \theta$.

Case IV: Unknown parameters α, β, θ.

Equating the third population moment to the third sample moment gives the equation

$$\sum_{x=0}^{\infty} x^3 \left[\theta^{(x)^{\beta} e^{\alpha(x)}} - \theta^{(x+1)^{\beta} e^{\alpha(x+1)}}\right] = \frac{1}{n}\sum_{i=1}^{n} x_i^3 \ . \qquad (4.120)$$

Solving the Equations (4.118) - (4.120) gives the estimators $\tilde{\alpha}, \tilde{\beta}, and\ \tilde{\theta}$ of the parameters $\alpha, \beta, and\ \theta$.

(3) The Maximum likelihood Method

The likelihood and log likelihood functions of the DMW (II) (α, β, θ) distribution are:

$$L = \prod_{i=1}^{n}\left[\theta^{x_i^{\beta} e^{\alpha x_i}} - \theta^{(x_i+1)^{\beta} e^{\alpha(x_i+1)}}\right],$$

$$\ln L = \sum_{i=1}^{n} \ln\left[\theta^{x_i^{\beta} e^{\alpha x_i}} - \theta^{(x_i+1)^{\beta} e^{\alpha(x_i+1)}}\right].$$

Case I: known parameters $\alpha\ and\ \beta$ and unknown parameter θ.

$$\frac{\partial\ lnL}{\partial\theta}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{(x_i)^{\beta} e^{\alpha(x_i)} \theta^{(x_i)^{\beta} e^{\alpha(x_i)}-1} - (x_i+1)^{\beta} e^{\alpha(x_i+1)} \theta^{(x_i+1)^{\beta} e^{\alpha(x_i+1)}-1}}{\left[\theta^{(x_i)^{\beta} e^{\alpha(x_i)}} - \theta^{(x_i+1)^{\beta} e^{\alpha(x_i+1)}}\right]} = 0\ . \qquad (4.121)$$

Solving the Equation (4.121) analytically gives the estimator $\hat{\theta}$ of the parameter θ.

Case II: known parameter β and unknown parameters α and θ.

$$\frac{\partial\, lnL}{\partial \alpha}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{\theta^{(x_i)^{\beta} e^{\alpha(x_i)}} (x_i)^{\beta}(x_i)e^{\alpha(x_i)} - \theta^{(x_i+1)^{\beta} e^{\alpha(x_i+1)}} (x_i+1)^{\beta}(x_i+1)e^{\alpha(x_i+1)}}{\left[\theta^{(x_i)^{\beta} e^{\alpha(x_i)}} - \theta^{(x_i+1)^{\beta} e^{\alpha(x_i+1)}}\right]} = 0. \tag{4.122}$$

Solving the Equations (4.121) and (4.122) gives the estimators $\hat{\alpha}\ and\ \hat{\theta}$ of the parameters $\alpha\ and\ \theta$.

Case III: known parameter α and unknown parameters $\beta\ and\ \theta$.

$$\frac{\partial\, lnL}{\partial \beta}|_{\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta}} = 0 \xrightarrow{yields}$$

$$\sum_{i=1}^{n} \frac{\theta^{(x_i)^{\beta} e^{\alpha(x_i)}} e^{\alpha(x_i)}(x_i)^{\beta} \ln(x_i) - \theta^{(x_i+1)^{\beta} e^{\alpha(x_i+1)}} e^{\alpha(x_i+1)}(x_i+1)^{\beta} \ln(x_i+1)}{\left[\theta^{(x_i)^{\beta} e^{\alpha(x_i)}} - \theta^{(x_i+1)^{\beta} e^{\alpha(x_i+1)}}\right]} = 0. \tag{4.123}$$

Solving the Equations (4.121), (4.123) gives the estimators $\hat{\beta}\ and\ \hat{\theta}$ of the parameters $\beta\ and\ \theta$.

Case IV: unknown parameters $\alpha, \beta, and\ \theta$.

Solving the Equations (4.121) - (4.123) gives the estimators $\hat{\alpha}, \hat{\beta}, and\ \hat{\theta}$ of the parameters $\alpha, \beta, and\ \theta$.

4.5.4 Special Distributions from DMW (II) Distribution

Many discretized distributions follow as special cases from DMW (II) distribution. Examples of such distributions are the discrete Weibull distribution, discrete extreme - value distribution, discrete Beta integrated model, and discrete Rayleigh distribution.

The DMW (II) (α, β, θ) distribution reduces to the discrete Weibull distribution DW (θ, β) distribution when $\alpha = 0$, where

$$S_1(x) = \theta^{(x+1)^\beta}, \theta = e^{-\lambda}.$$

(Nakagawa and Osaki (1975))

When $\beta = 2, \alpha = 0$ the DMW (II) (α, β, θ) distribution reduces to the discrete Rayleigh DR (θ) distribution, where,

$$S_1(x) = \theta^{(x+1)^2}, \theta = e^{-\lambda}.$$

(Dilip Roy (2004))

Also, the DMW (II) (α, β, θ) distribution reduces to the discrete Extreme-value Type (I) DEXV (I) (α, θ) distribution by setting $\beta = 0$, where

$$S_1(x) = \theta^{e^{\alpha(x+1)}}, \theta = e^{-\lambda}.$$

(Nakagawa (1978))

4.6 Summary

In this chapter, many issues regarding the discrete general class of continuous distributions theoretically and could be summarized as below:

1) Generating discrete form of the general class of continuous distributions
2) Deducing reliability measures and characteristics of the discrete general class.
3) Introducing some generalized discrete models, such as the DMWE, DMW (I) and DMW (II).
4) Obtaining new discrete distributions obtained from the DMWE and DMW (I) which are called DChen and DLFR distributions respectively.

5) Discussing the probability mass function, the cumulative distribution function and some of their distributional properties.
6) However, the closed form for the distributional properties cannot be achieved.
7) Estimating the parameters in several cases by using the PM, MM and MLM.
8) The unknown parameter θ has a proportion estimator in exact solution.

Chapter V

Simulation Studies

5.1 Introduction

A MathCAD simulation is an environment for Computational Thinking – an approach to calculation, data analysis and problem solving that uses the capabilities of a computer to construct better solutions. This program is used frequently in our study to conduct simulation studies and to get numerical results.

Some discrete distributions are introduced from discretizing a general class of continuous distributions. These are DMWE, DChen, DMW (I), DLFR and DMW (II) distributions. The properties of these distributions are studied. The parameters of these distributions are estimated. The estimation methods are compared based on the performance of the estimators.

5.2 The DMWE Distribution

The properties of the DMWE (α, β, θ) distribution are studied. The performance of the parameters at selected values of parameters with different sample sizes and different cases are compared.

5.2.1 Properties of the DMWE Distribution

The properties of the DMWE distribution are studied, such as the r-th moments, the central moments, first and second moments and mode at selected values of parameters.

Using Equation (4.27) the first four moments of the DMWE (α, β, θ) distribution are determined for some selected values of the parameters. The results are included in Table 5.1.

Table 5.1 The r^{th} moments at different values of parameters

	$\alpha = 2, \theta = 0.67$			$\alpha = 3, \theta = 0.549$			$\alpha = 4, \theta = 0.449$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	1.611	1.428	1.451	2.001	2.056	2.151	2.28	2.595	2.804
2	4.168	2.567	2.456	6.561	5.229	5.222	8.695	8.291	8.808
3	12.601	5.045	4.466	25.615	14.664	13.561	40.085	29.333	29.587
4	42.488	10.601	8.487	112.627	44.071	36.893	209.869	111.385	104.279
	$\alpha = 2, \theta = 0.368$			$\alpha = 3, \theta = 0.223$			$\alpha = 4, \theta = 0.135$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	0.735	0.932	1.055	0.8886	1.347	0.0001	0.986	1.698	2.117
2	1.227	1.292	1.414	1.776	2.518	1.41	2.216	3.909	5.239
3	2.427	2.013	2.132	4.345	5.326	4.506	6.202	10.249	14.159
4	5.516	3.458	3.567	12.382	12.358	23.769	20.441	29.504	70.729
	$\alpha = 2, \theta = 0.018$			$\alpha = 3, \theta = 0.0024$			$\alpha = 4, \theta = 0.00033$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	0.075	0.32	0.587	0.097	0.529	0.924	0.109	0.702	1.241
2	0.77	0.323	0.589	0.103	0.599	1.176	0.12	0.918	1.991
3	0.081	0.327	0.593	0.118	0.738	1.682	0.142	1.619	3.978
4	0.089	0.335	0.601	0.146	1.018	2.694	0.195	2.298	7.028

Table 5.1 shows that when θ decreases and α increases, the r^{th} moments increase for most values of β. For fixed α and θ when β becomes large, the r^{th} moments decrease.

Using the results in Table 5.1 central moments, skewness, and kurtosis are included in Tables 5.2.

Table 5.2 The central moment, skewness and kurtosis at different values of parameters

	$\alpha = 2, \theta = 0.67$			$\alpha = 3, \theta = 0.549$			$\alpha = 4, \theta = 0.449$		
r	β 1	2	3	1	2	3	1	2	3
2	1.573	0.527	0.351	2.551	1.001	0.597	3.497	1.556	0.947
3	0.819	-0.126	-0.116	2.257	-0.204	-0.24	4.313	-0.261	-0.417
4	5.984	0.714	0.294	17.118	2.483	0.986	34.436	5.853	2.498
α_3	0.415	-0.329	-0.558	0.552	-0.204	-0.52	0.66	-0.134	-0.452
α_4	2.418	2.571	2.386	2.001	2.478	2.766	2.816	2.417	2.785
	$\alpha = 2, \theta = 0.368$			$\alpha = 3, \theta = 0.223$			$\alpha = 4, \theta = 0.135$		
r	β 1	2	3	1	2	3	1	2	3
2	0.687	0.423	0.301	1.756	0.929	1.525	1.244	1.025	0.759
3	0.515	0.02	0.0005	3.683	0.739	-1.548	1.564	0.129	-0.145
4	1.482	0.424	0.298	12.008	2.089	3.993	6.07	2.573	1.536
α_3	0.904	0.073	0.03	1.583	0.825	-0.804	1.127	0.124	-0.219
α_4	3.14	2.37	3.289	3.894	2.421	1.717	3.922	2.449	2.666
	$\alpha = 2, \theta = 0.018$			$\alpha = 3, \theta = 0.0024$			$\alpha = 4, \theta = 0.00033$		
r	β 1	2	3	1	2	3	1	2	3
2	0.071	0.22	0.244	0.094	0.319	0.323	0.109	0.425	0.45
3	0.064	0.083	-0.04	0.089	0.084	-0.00019	0.108	0.123	-0.000785
4	0.067	0.083	0.07	0.106	0.229	0.318	0.141	0.454	0.536
α_3	13.291	0.804	-0.332	3.088	0.466	-0.011	3.001	0.444	-0.026
α_4	3.14	1.715	1.176	11.996	2.25	3.048	11.87	2.513	2.647

Table 5.2 shows that the central moments decrease when β becomes large at fixed values of α and θ decreases. For small values of β and θ, and α large, the distribution becomes positively skewed and leptokurtic, but when β becomes large, the distribution becomes negatively skewed and platykurtic.

The mode of the DMWE distribution is determined for different values of the parameters. The results are included in Table 5.3.

Table 5.3 The mode of the DMWE distribution at different values of parameters

$\alpha = 2, \theta = 0.76$				$\alpha = 3, \theta = 0.549$				$\alpha = 4, \theta = 0.449$			
β				β				β			
1	2	3	≥ 4	1	2	3	≥ 4	1	2	3	≥ 4
1	2	2	2	1	2	2	2	0	3	3	3
$\alpha = 2, \theta = 0.368$				$\alpha = 3, \theta = 0.223$				$\alpha = 4, \theta = 0.135$			
β				β				β			
1	2	3	≥ 4	1	2	3	≥ 4	1	2	3	≥ 4
0	1	1	1	0	1	2	2	0	2	2	3
$\alpha = 2, \theta = 0.018$				$\alpha = 3, \theta = 0.00247$				$\alpha = 4, \theta = 0.00033$			
β				β				β			
1	2	3	≥ 4	1	2	3	≥ 9	1	2	3	≥ 9
0	0	1	1	0	0	1	2	0	1	1	3

Table 5.3 shows that when α, β increase and θ decreases, the mode increases. The value of mode does not change when β ≥ 4, except for too small values of θ, the mode does not change when β ≥ 9.

5.2.2 Performance of Estimators of DMWE Parameters

The parameters of the DMWE distribution are estimated by the proportion method, the moments method, and the maximum likelihood method. This is done in four different cases:

Case I: known parameters α *and* β and unknown parameter θ

The performance of the proportion estimator θ^*, the method-of-moments estimator $\tilde{\theta}$, and the maximum likelihood estimator $\hat{\theta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.4.

Table 5.4 Case I: known parameters $\alpha = 3$ *and* $\beta = 5$ and unknown parameter θ

		$\theta = 0.105$		
λ	0.75	PM	MM	MLM
n		θ^*	$\tilde{\theta}$	$\hat{\theta}$
10	Bias Var MSE	0.805 0.082 0.729	0.022 0.0096 0.01	0.019 0.0057 0.0060
20	Bias Var MSE	0.715 0.148 0.658	0.013 0.0056 0.0057	0.016 0.0014 0.0017
50	Bias Var MSE	0.497 0.237 0.484	0.0064 0.0023 0.0024	0.0035 0.00097 0.00098
60	Bias Var MSE	0.48 0.238 0.468	0.0048 0.0020 0.0021	0.00309 0.00070 0.00071
100	Bias Var MSE	0.043 0.009 0.011	0.00014 0.00016 0.00016	0.000429 0.0011 0.0011
200	Bias Var MSE	0.01 0.001 0.001	−0.000187 0.0000243 0.0000243	−0.00078 0.00081 0.00081

From Table 5.4 it is observed that for the estimation of θ when α and β are known, the MM is the best when the sample size becomes large. When the effective sample size is small (say less than 30) or moderate then the MLM is the best one to be applied.

Case II: known parameter β and unknown parameters α and θ

The performance of the proportion estimators $\alpha^*$and θ^*, the method-of-moment estimators $\tilde{\alpha}$ and $\tilde{\theta}$, and the maximum likelihood estimators $\hat{\alpha}, \hat{\theta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.5.

Table 5.5 Case II: known parameter $\beta = 5$ and unknown parameters θ, α

		$\theta = 0.105, \alpha = 3$					
λ	0.75	PM		MM		MLM	
n		θ^*	α^*	$\tilde{\theta}$	$\tilde{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
10	Bias Var MSE	0.359 0.249 0.377	-0.568 1.09 1.412	-0.025 0.012 0.013	0.812 2.93 3.588	0.12 0.0091 0.024	-0.105 0.056 0.067
20	Bias Var MSE	0.205 0.214 0.256	-0.287 0.978 1.06	-0.018 0.0048 0.0051	0.759 2.241 2.816	0.056 0.012 0.016	0.05 0.165 0.168
50	Bias Var MSE	0.129 0.179 0.196	-0.068 0.47 0.474	−0.0053 0.0054 0.0054	0.141 0.071 0.092	0.021 0.012 0.012	0.073 0.076 0.081
60	Bias Var MSE	0.123 0.176 0.191	-0.059 0.945 0.948	−0.00517 0.0038 0.0038	0.159 0.208 0.234	0.018 0.011 0.011	0.144 0.122 0.142
100	Bias Var MSE	0.0019 0.00036 0.00036	-0.023 1.203 1.203	−0.0009 0.0042 0.0042	0.14 0.142 0.161	0.0001 0.026 0.026	0.249 0.263 0.325
200	Bias Var MSE	0.00136 0.00036 0.00038	−0.00007 0.0013 0.0013	−0.00063 0.0044 0.0044	0.0023 0.0016 0.0016	0.00006 0.01 0.01	0.189 0.152 0.188

Table 5.5 shows the results for the estimation of θ *and* α when $\beta = 5$. For the estimation of θ, the MM is better than the PM and the MLM for different sample sizes. For the estimation of α, the MLM is the best when the effective sample sizes are small (say, less than 30) or moderate. When the sample sizes become large the PM is the best one to be applied.

Case III: known parameter α and unknown parameters β *and* θ

The performance of the proportion estimators β^* *and* θ^*, the method-of-moments estimators $\tilde{\beta}$ and $\tilde{\theta}$, and the maximum likelihood estimators $\hat{\beta}$ *and* $\hat{\theta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.6.

Table 5.6 Case III: known parameter $\alpha = 3$ and unknown parameters θ, β

		$\theta = 0.105$, $\beta = 4$					
λ	0.75	PM		MM		MLM	
n		θ^*	β^*	$\tilde{\theta}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\beta}$
10	Bias Var MSE	-0.09 0.003 0.011	4.328 13.983 32.717	0.419 0.166 0.342	0.432 0.971 1.157	0.161 0.144 0.17	-0.446 23.479 23.677
20	Bias Var MSE	-0.03 0.016 0.017	3.976 17.779 33.592	0.25 0.144 0.206	0.324 0.544 0.649	0.071 0.075 0.08	0.083 8.225 8.231
50	Bias Var MSE	-0.003 0.012 0.012	1.466 11.331 13.479	0.0027 0.0001307 0.0001381	0.013 0.018 0.018	0.011 0.0017 0.0018	0.364 0.249 0.382
60	Bias Var MSE	-0.002 0.011 0.011	1.084 8.278 9.453	0.0024 0.0001676 0.0001734	0.047 0.021 0.023	0.0091 0.0074 0.0075	0.868 2.871 3.624
100	Bias Var MSE	-0.001 0.002 0.002	0.078 0.21 0.216	0.000035 0.000095 0.000095	0.0022 0.018 0.018	0.0004 0.0035 0.0035	0.472 1.243 1.465
200	Bias Var MSE	−0.00008 0.002 0.002	0.049 0.223 0.225	0.000020 0.000108 0.000108	0.013 0.012 0.012	−0.00056 0.0025 0.0025	0.717 0.808 1.323

Table 5.6 represents the results for the estimation of $\theta \, and \, \beta$ when $\alpha = 3$. For the estimation of θ, the MM is better than the PM and MLM. When the sample sizes become small (say, less than 30) the PM will be the best. For the estimation of β, the MM is always better than the PM and MLM for different sample sizes.

Case IV: unknown parameters θ , α, and β

The performance of the proportion estimators $\alpha^*, \beta^*, \theta^*$, the method-of-moments estimators $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}$, and the maximum likelihood estimators $\hat{\alpha}, \hat{\beta}, \hat{\theta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.7.

Table 5.7 Case IV: unknown parameters θ , α, and β

		$\theta = 0.105, \alpha = 3, \beta = 4$								
λ	0.75	PM			MM			MLM		
n		θ^*	α^*	β^*	$\tilde{\theta}$	$\tilde{\alpha}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
10	Bias	-0.08	0.21	2.842	0.342	-0.354	-0.245	-0.035	0.21	2.157
	Var	0.02	0.472	54.01	0.123	1.801	1.919	0.0075	0.971	2.112
	MSE	0.027	0.516	62.087	0.24	1.926	1.979	0.0087	1.015	6.764
20	Bias	-0.072	0.209	2.564	0.262	0.438	-0.151	-0.023	0.03	1.8
	Var	0.025	0.507	48.433	0.149	2.823	1.434	0.0068	1.047	1.863
	MSE	0.03	0.55	55.005	0.218	3.015	1.457	0.0073	1.048	5.103
50	Bias	-0.004	0.054	0.486	0.082	0.617	0.292	−0.0055	-0.275	1.787
	Var	0.064	0.389	11.975	0.082	1.751	0.896	0.0066	0.786	3.447
	MSE	0.064	0.39	12.211	0.088	2.132	0.981	0.0066	0.862	6.642
60	Bias	-0.003	0.035	0.186	0.0048	0.07	0.02	−0.0051	-0.221	1.57
	Var	0.064	0.299	5.312	0.0071	0.062	0.045	0.0067	0.901	2.278
	MSE	0.064	0.3	5.347	0.0071	0.067	0.045	0.0067	0.95	4.743
100	Bias	-0.001	0.023	0.112	0.0014	0.061	0.068	−0.0027	-0.274	1.959
	Var	0.064	0.304	5.745	0.0046	0.049	0.066	0.0071	0.795	3.616
	MSE	0.064	0.304	5.757	0.0046	0.053	0.07	0.0071	0.87	7.455
200	Bias	-0.001	0.021	0.037	0.00046	0.045	0.062	0.0022	-0.187	1.632
	Var	0.062	0.322	3.432	0.0048	0.033	0.048	0.0074	0.978	2.073
	MSE	0.062	0.323	3.434	0.0048	0.035	0.052	0.0074	1.013	4.737

From Table 5.7 it is observed that for the estimation of θ, the MLM is better than the PM and the MM when the sample sizes are small (less than 30) or moderate. When the sample sizes become large, the MM will be the best one. For the estimation of α, the PM is the best method. For the estimation of β, the MM is always better than the PM and MLM for different sample sizes.

5.3 The DChen Distribution

The properties of the DChen (β, θ) distribution are studied. The performance of the parameters by three methods of estimation at selected values of parameters with different sample sizes and different cases is discussed.

5.3.1 Properties of the DChen Distribution

The properties of the DChen (β, θ) distribution are discussed, such as the r^{th} moments, the central moments, first and second moments and mode at selected values of parameters. The first four moments of the DChen (β, θ) distribution are determined for some selected values of the parameters. The results are included in Table 5.8

Table 5.8 The r^{th} moments at different values of parameters

	$\theta = 0.951$			$\theta = 0.905$			$\theta = 0.779$		
r	β			β			B		
	0.5	1	2	0.5	1	2	0.5	1	2
1	3.569	2.099	0.98	3.535	1.52	0.847	1.875	0.86	0.651
2	24.648	5.508	1.12	21.639	3.2	0.856	7.756	1.3	0.651
3	190.94	15.896	1.39	154.36	7.529	0.875	40.752	2.22	0.651
4	1579	49.079	1.94	1203	19.22	0.913	249.96	4.24	0.651

Table 5.8 shows that the r^{th} moments decrease when β increases and θ decreases.

For different values of parameters, central moments, skewness, and kurtosis calculated. The results are included in Table 5.9.

Table 5.9 The central moment, skewness and kurtosis at different values of parameters

	$\theta = 0.951$			$\theta = 0.905$			$\theta = 0.779$		
r	β			β			β		
	0.5	1	2	0.5	1	2	0.5	1	2
2	8.531	1.105	0.151	8.263	0.881	0.139	4.235	0.558	0.227
3	30.017	-0.3	-0.007	16.337	-0.027	-0.086	10.319	0.147	-0.069
4	207.129	3.017	0.15	163.214	1.753	0.09	70.806	0.696	0.072
α_3	1.2	-0.25	-0.12	0.68	-0.033	-1.72	1.18	0.35	-0.63
α_4	2.84	2.47	6.52	2.39	2.26	4.74	3.94	2.23	1.44

Table 5.9 shows that the most central moments decrease when β increases and θ decreases. The distribution becomes positively skewed when β small. For large values of β the distribution becomes negatively skewed. The distribution is platykurtic for small values of β and becomes leptokurtic when β large.

The mode of the DChen distribution is determined for different values of the parameters. The results are included in Table 5.10.

Table 5.10 The mode at different values of parameters

$\theta = 0.951$				$\theta = 0.905$				$\theta = 0.779$			
β				β				β			
0.5	0.7	1	≥ 2	0.5	0.7	1	≥ 2	0.5	0.7	1	≥ 2
0	4	2	1	0	2	2	1	0	0	1	1

Table 5.10 represents the values of mode when β increases and θ decreases. The mode does not change when β ≥ 2.

5.3.2 Performance of Estimators of DChen Parameters

The parameters of the DChen distribution are estimated by the proportion method, the moments method, and the maximum likelihood method.

The performance of estimators of the parameters β, θ are presented by using three methods of estimation such as bias, variance and mean squared error at selected values of parameters with different sample sizes in two cases.

Case I: known parameter $\boldsymbol{\beta}$ and unknown parameter $\boldsymbol{\theta}$.

The performance of the proportion estimator θ^*, the method-of-moments estimator $\tilde{\theta}$, and the maximum likelihood estimator $\hat{\theta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.11.

Table 5.11 Case I: known parameter $\boldsymbol{\beta = 0.65}$ and unknown parameter $\boldsymbol{\theta}$

		$\theta = 0.638$		
λ	0.45	PM	MM	MLM
n		θ^*	$\tilde{\theta}$	$\hat{\theta}$
10	Bias Var MSE	-0.005 0.008 0.008	-0.053 0.0060 0.0088	-0.038 0.019 0.021
20	Bias Var MSE	-0.004 0.008 0.008	-0.022 0.00031 0.00081	-0.024 0.008614 0.009187
50	Bias Var MSE	-0.003 0.018 0.018	-0.0079 0.00035 0.00042	-0.0093 0.00109 0.00117
60	Bias Var MSE	-0.002 0.002 0.002	-0.0032 0.0020 0.0021	-0.0051 0.00047 0.0005
100	Bias Var MSE	-0.001 0.001 0.001	-0.00015 0.000038 0.000038	-0.00029 0.0012 0.0012
200	Bias Var MSE	0.000051 0.003 0.003	0.00042 0.000059 0.00005	0.00057 0.000056 0.000056

Table 5.11 represents the results for the estimation of θ when $\beta = 0.65$ is known. The PM is better than the MM and the MLM when the sample sizes become small (say, less than 30) and large. The MM performs slightly better than the MLM when the sample sizes become moderate.

Case II: unknown parameters $\boldsymbol{\theta}$ *and* $\boldsymbol{\beta}$

The performance of the proportion estimators $\theta^* and\ \beta^*$, the method-of-moments estimators $\tilde{\theta} and\ \tilde{\beta}$, and the maximum likelihood estimator $\hat{\theta}\ and\ \hat{\beta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.12.

Table 5.12 Case II: unknown parameters θ *and* β

		$\theta = 0.638, \beta = 0.65$					
λ	0.45	PM		MM		MLM	
n		θ^*	β^*	$\tilde{\theta}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\beta}$
10	Bias Var MSE	0.018 0.004 0.004	0.091 0.014 0.022	-0.078 0.0092 0.015	0.167 0.348 0.375	0.122 0.097 0.112	0.185 0.122 0.156
20	Bias Var MSE	0.008 0.016 0.016	0.046 0.041 0.043	-0.035 0.016 0.017	−0.0039 0.019 0.019	0.102 0.115 0.126	0.151 0.14 0.162
50	Bias Var MSE	0.002 0.002 0.002	0.004 0.015 0.015	-0.01 0.00084 0.00094	−0.0034 0.00083 0.00084	0.061 0.124 0.128	0.103 0.154 0.164
60	Bias Var MSE	0.002 0.003 0.003	0.008 0.018 0.018	−0.0062 0.0023 0.0023	−0.0079 0.0039 0.0039	0.031 0.159 0.16	0.035 0.56 0.561
100	Bias Var MSE	0.00026 0.003 0.003	0.002 0.018 0.018	0.00031 0.0025 0.0025	0.024 0.0052 0.0058	0.002 0.001 0.001	0.0071 0.306 0.306
200	Bias Var MSE	0.00026 0.002 0.002	0.00008 0.028 0.028	0.00047 0.0076 0.0076	0.00091 0.0049 0.0049	0.001 0.001 0.001	−0.00028 0.168 0.168

Table 5.12 shows the results for the estimation of θ *and* β when they are unknown. For the estimation of θ, the PM is better than the MM and the MLM when the sample sizes become small (say, less than 30) or large. When the sample sizes are moderate, the MM is the best method. For the estimation of β, the PM is the best for different sample sizes.

5.4 The DMW (I) Distribution

The properties of the DMW (I) (α, β, θ) distribution are studied. The performance of the parameters by three methods of estimation at selected values of parameters with different sample sizes and different cases are discussed.

5.4.1 Properties of the DMW (I) Distribution

The properties of the DMW (I) distribution are studied. These properties include the r^{th} moments, the central moments, first and second moments and mode at selected values of parameters.

The first four moments of the DMW (I) are determined for different values of the parameters. The results are included in Table 5.13.

Table 5.13 The r^{th} moments at different values of parameters

	$\alpha = 3, \theta = 0.951$			$\alpha = 1.5, \theta = 0.905$			$\alpha = 4, \theta = 0.449$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	2.797	2.325	1.504	2.347	1.376	0.975	1.083	0.63	0.536
2	15.878	8.991	3.3	11.708	3.259	1.466	3.362	0.86	0.563
3	109.594	43.129	8.32	74.181	9.447	2.515	14.61	1.365	0.617
4	842.58	240.485	23.269	538.489	31.915	4.816	80.194	2.519	0.726
	$\alpha = 10, \theta = 0.951$			$\alpha = 5, \theta = 0.905$			$\alpha = 1, \theta = 0.607$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	1.335	1.116	0.887	1.2	0.922	0.729	0.582	0.42	0.375
2	4.716	2.968	1.646	3.969	2.013	1.121	1.257	0.53	0.388
3	22.976	10.324	3.616	18.265	5.579	1.999	3.77	0.765	0.415
4	138.042	43.866	9.06	104.966	18.634	4.042	14.858	1.284	0.469
	$\alpha = 15, \theta = 0.951$			$\alpha = 7.5, \theta = 0.905$			$\alpha = 1.5, \theta = 0.607$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	0.814	0.731	0.628	0.745	0.632	0.535	0.402	0.318	0.291
2	2.126	1.579	1.049	1.849	1.182	0.764	0.724	0.383	0.299
3	7.813	4.49	2.093	6.44	2.812	1.267	1.756	0.521	0.315
4	37.178	15.92	4.841	29.209	8.183	2.406	5.595	0.819	0.348

Table 5.13 shows that the r^{th} moments decrease when α, β increase and θ decreases.

The central moments, the skewness, and the kurtosis are determined for different values of the parameters. The results are included in Table 5.14.

Table 5.14 The central moment, skewness and kurtosis at different values of parameters

	$\alpha = 3, \theta = 0.951$			$\alpha = 1.5, \theta = 0.905$			$\alpha = 0.3, \theta = 0.607$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
2	7.186	3.584	1.038	6.665	1.969	0.663	2.188	0.463	0.276
3	22.546	5.556	0.234	20.031	2.155	0.116	6.23	0.241	0.02
4	171.314	43.339	2.656	154.288	13.121	1.088	36.439	0.654	0.125
α_3	1.17	0.818	0.221	1.164	0.779	0.215	1.925	0.765	0.138
α_4	3.317	3.374	2.466	3.473	3.384	2.478	7.612	3.056	1.645
	$\alpha = 10, \theta = 0.951$			$\alpha = 5, \theta = 0.905$			$\alpha = 1, \theta = 0.607$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
2	2.93	1.722	0.859	2.527	1.163	0.589	0.919	0.353	0.248
3	8.855	3.166	0.632	7.436	1.58	0.322	1.97	0.246	0.084
4	56.257	15.298	2.143	45.372	6.158	0.94	8.294	0.466	0.115
α_3	1.766	1.402	0.794	1.851	1.259	0.712	2.236	1.177	0.677
α_4	6.553	5.159	2.904	7.105	4.551	2.708	9.815	3.728	1.855
	$\alpha = 15, \theta = 0.951$			$\alpha = 7.5, \theta = 0.905$			$\alpha = 1.5, \theta = 0.607$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
2	1.463	1.045	0.655	1.293	0.782	0.478	0.563	0.282	0.214
3	3.699	1.81	0.612	3.314	1.075	0.347	1.013	0.22	0.104
4	18.871	6.997	1.598	15.243	3.428	0.762	3.397	0.358	0.112
α_3	2.089	1.695	1.155	2.132	1.553	1.052	2.4	1.476	1.061
α_4	8.818	6.407	3.725	9.116	5.601	3.342	10.716	4.532	2.435

Table 5.14 represents that the central moments decrease when β increases and the values of α and θ decrease. The distribution becomes positively skew for different values of α, β and θ. The distribution becomes leptokurtic when α and θ decrease and β small. The distribution becomes platykurtic when β large. For

different values of the parameters, the mode is calculated. The mode values are included in Table 5.15.

Table 5.15 The mode at different values of parameters

$\alpha = 3, \theta = 0.951$				$\alpha = 1.5, \theta = 0.905$				$\alpha = 0.3, \theta = 0.607$			
β				β				β			
1	2	3	≥ 4	1	2	3	≥ 4	1	2	3	≥ 4
0	1	2	1	0	1	1	1	0	0	1	1
$\alpha = 10, \theta = 0.951$				$\alpha = 5, \theta = 0.905$				$\alpha = 1, \theta = 0.607$			
β				β				β			
1	2	3	≥ 5	1	2	3	≥ 4	1	2	3	≥ 4
0	0	0	1	0	0	0	1	0	0	0	0
$\alpha = 15, \theta = 0.951$				$\alpha = 7.5, \theta = 0.905$				$\alpha = 1.5, \theta = 0.607$			
β				β				β			
1	2	3	≥ 4	1	2	3	≥ 4	1	2	3	≥ 4
0	0	0	0	0	0	0	0	0	0	0	0

5.4.2 Performance of Estimators of DMW (I) Parameters

The parameters of the DMW (I) distribution are estimated by the proportion method, the moments method, and the maximum likelihood method.

The performance of estimators of the parameters α, β, θ are presented by using three methods of estimation such as bias, variance and mean squared error at selected values of parameters with different sample sizes in four cases.

Case I: known parameters $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ and unknown parameter $\boldsymbol{\theta}$.

The performance of the proportion estimator θ^*, the method-of-moments estimator $\tilde{\theta}$, and the maximum likelihood estimator $\hat{\theta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.16.

Table 5.16 Case I:known parameters $\alpha = 0.2, \beta = 5$ and unknown parameter θ

		$\theta = 0.472$		
α_1	0.15	PM	MM	MLM
n		θ^*	$\tilde{\theta}$	$\hat{\theta}$
10	Bias Var MSE	0.024 0.008 0.009	0.096 0.01 0.019	-0.011 0.015 0.015
20	Bias Var MSE	0.02 0.001 0.002	0.045 0.006 0.008	-0.005 0.012 0.012
50	Bias Var MSE	0.006 0.004 0.004	0.006 0.006 0.006	-0.002 0.003 0.003
60	Bias Var MSE	0.006 0.001 0.001	0.003 0.002 0.002	-0.001 0.006 0.006
100	Bias Var MSE	0.005 0.001 0.001	-0.00045 0.004 0.004	-0.000042 0.002 0.002
200	Bias Var MSE	0.001 0.001 0.001	-0.00042 0.001 0.001	-0.000032 0.004 0.004

From Table 5.16 for the estimation of , it seems that the PM is better than the MM and the MLM only when the sample sizes become small (less than 30). When the sample sizes are moderate or large then the ML method is the best one to apply.

Case II: known parameter β ($\beta = 5$) and unknown parameters θ and α

The performance of the proportion estimators θ^* and α^*, the method-of-moments estimators $\tilde{\theta}$ and $\tilde{\alpha}$, and the maximum likelihood estimators $\hat{\theta}$ and $\hat{\alpha}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.17.

Table 5.17 Case II: known parameter $\boldsymbol{\beta} = 5$ and unknown parameters $\boldsymbol{\theta}, \boldsymbol{\alpha}$

		$\theta = 0.472, \alpha = 0.2$					
α_1	0.15	PM		MM		MLM	
n		θ^*	α^*	$\tilde{\theta}$	$\tilde{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
10	Bias Var MSE	-0.034 0.015 0.016	-0.068 0.065 0.07	-0.012 0.001 0.002	-0.015 0.002 0.002	-0.021 0.011 0.012	0.053 0.936 0.939
20	Bias Var MSE	-0.016 0.008 0.009	-0.042 0.007 0.009	-0.01 0.002 0.002	-0.038 0.006 0.007	-0.015 0.007 0.007	0.005 0.04 0.04
50	Bias Var MSE	-0.002 0.0000966 0.0001018	-0.006 0.054 0.054	-0.003 0.001 0.001	-0.007 0.001 0.001	-0.003 0.004 0.004	-0.013 0.039 0.039
60	Bias Var MSE	-0.002 0.000036 0.000039	-0.006 0.041 0.041	-0.003 0.001 0.001	-0.002 0.001 0.001	-0.003 0.004 0.004	0.005 0.032 0.032
100	Bias Var MSE	-0.001 0.000085 0.000086	-0.003 0.028 0.028	-0.001 0.00036 0.00036	-0.00047 0.001 0.001	-0.001 0.003 0.003	-0.004 0.026 0.026
200	Bias Var MSE	-0.001 0.000036 0.000037	-0.001 0.038 0.038	-0.0004 0.001 0.001	-0.014 0.002 0.002	-0.0004 0.003 0.003	0.01 0.075 0.075

Table 5.17 represents the results for the estimation of θ *and* α when $\beta = 5$. For the estimation of θ, the PM is better than the MM and the MLM when the sample sizes are moderate or large. When the sample sizes are small (say, less than 30) the MM is the best. For the estimation of α, the MM is the best when the sample sizes are small (say, less than 30) or moderate. When the sample sizes become large the PM is the best one.

Case III: known parameter $\boldsymbol{\alpha = 0.2}$ and unknown parameters $\boldsymbol{\theta}$ *and* $\boldsymbol{\beta}$

The performance of the proportion estimators θ^* *and* β^*, the method-of-moments estimators $\tilde{\theta}$ *and* $\tilde{\beta}$, and the maximum likelihood estimators $\hat{\theta}$ *and* $\hat{\beta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.18.

Table 5.18 Case III: known parameter $\alpha = 0.2$ and unknown parameters θ, β

		$\theta = 0.472, \beta = 5$					
α_1	0.15	PM		MM		MLM	
n		θ^*	β^*	$\tilde{\theta}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\beta}$
10	Bias Var MSE	0.006 0.011 0.011	0.121 0.192 0.206	0.026 0.031 0.031	0.279 0.727 0.805	0.336 0.081 0.194	1.31 23.025 24.771
20	Bias Var MSE	0.005 0.008 0.008	0.112 0.149 0.162	0.015 0.009 0.009	0.167 0.16 0.188	0.317 0.085 0.185	1.03 24.397 25.458
50	Bias Var MSE	0.001 0.006 0.006	0.085 0.094 0.101	0.004 0.006 0.006	0.12 0.103 0.117	0.293 0.089 0.175	0.659 25.866 26.301
60	Bias Var MSE	0.001 0.005 0.005	0.085 0.085 0.093	0.002 0.002 0.002	0.123 0.042 0.057	0.282 0.089 0.169	0.527 26.57 26.847
100	Bias Var MSE	−0.000087 0.007 0.007	0.053 0.109 0.111	0.00011 0.001 0.001	0.115 0.022 0.035	0.225 0.095 0.145	-0.533 27.176 27.46
200	Bias Var MSE	−0.000084 0.007 0.007	0.041 0.265 0.267	0.0000087 0.004 0.004	0.111 0.079 0.091	0.206 0.095 0.137	-0.86 27.059 27.798

Table 5.18 Shows the results for the estimation of θ *and* β when $\alpha = 0.2$. For the estimation of θ *and* β, the PM is better than the MM and the MLM when the effective sample sizes are small (say, less than 30) or moderate. When the sample size is large then the MM will be the best for estimating θ *and* β.

Case IV: unknown parameters θ, α *and* β

The performance of the proportion estimators θ^*, α^* *and* β^*, the method-of-moments estimators $\tilde{\theta}$, $\tilde{\alpha}$ *and* $\tilde{\beta}$, and the maximum likelihood estimators $\hat{\theta}$,

$\hat{\alpha}$ *and* $\hat{\beta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.19.

Table 5.19 Case IV: unknown parameters $\boldsymbol{\theta}, \boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$

		$\theta = 0.472, \alpha = 0.2, \beta = 5$								
α_1	0.15	PM			MM			MLM		
n		θ^*	α^*	β^*	$\tilde{\theta}$	$\tilde{\alpha}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
10	Bias	0.027	0.102	1.588	-.012	-.015	0.149	0.526	18.446	1.511
	Var	0.021	0.026	0.431	0.005	0.003	0.094	.0000035	1542.74	33.931
	MSE	0.021	0.036	2.951	0.005	0.003	0.116	0.276	1882.98	36.215
20	Bias	0.016	0.027	0.473	-.009	-.005	0.155	0.526	18.234	1.593
	Var	0.012	0.001	0.115	0.003	0.001	0.044	.0000035	1509.85	33.525
	MSE	0.012	0.001	0.339	0.003	0.001	0.068	0.276	1842.32	36.063
50	Bias	0.005	0.014	0.275	-.003	-.008	0.177	0.525	17.607	1.304
	Var	0.002	0.002	0.049	0.002	0.001	0.041	.0000078	1417.25	37.213
	MSE	0.002	0.002	0.125	0.002	0.001	0.073	0.275	1727.25	38.912
60	Bias	0.004	0.011	0.272	-.003	-.013	0.18	0.525	17.61	1.228
	Var	0.002	0.002	0.012	0.003	0.002	0.046	.000007	1418.66	36.835
	MSE	0.002	0.002	0.086	0.003	0.002	0.078	0.275	1728.78	38.344
100	Bias	0.001	0.001	0.103	-.002	-.012	0.182	0.523	15.677	1.497
	Var	0.004	0.004	0.19	0.005	0.003	0.092	.000007	1138.43	35.06
	MSE	0.004	0.004	0.201	0.005	0.003	0.125	0.273	1384.16	37.302
200	Bias	−.00039	−.00003	0.089	-.001	-.013	0.194	0.52	18.54	1.602
	Var	0.004	0.001	0.116	0.004	0.001	0.057	.0000001	1558.14	33.185
	MSE	0.004	0.001	0.124	0.004	0.002	0.095	0.271	1901.78	35.752

Table 5.19 shows the results for the estimation of θ, α *and* β when they are unknown. For the estimation of θ *and* α , the MM is better than the PM and the ML method when the effective sample sizes are small (say, less than 30) or moderate. When the sample sizes become large, the PM will be the best method. For the estimation of β the MM is always the best method to apply.

5.5 The DLFR Distribution

The properties of the DLFR (α, θ) distribution are studied. The performance of the parameters by three methods of estimation at selected values of parameters with different sample sizes and different cases are discussed.

5.5.1 Properties of the DLFR Distribution

The properties of the DLFR distribution are studied. These properties include the r^{th} moments, the central moments, first and second moments and mode at selected values of parameters.

The first four moments of the DLFR are determined for different values of the parameters. The results are included in Table 5.20.

Table 5.20 The r^{th} moments at different values of parameters

	$\theta = 0.905$			$\theta = 0.779$			$\theta = 0.607$		
r	α			α			α		
	1.5	2.5	5	0.6	1	2	0.3	0.5	1
1	1.698	1.409	0.922	1.021	0.887	0.634	0.63	0.56	0.42
2	4.854	3.704	2.013	1.903	1.577	1.015	0.86	0.746	0.53
3	17.091	12.147	5.579	4.287	3.4	1.969	1.365	1.151	0.765
4	70.176	46.952	18.634	11.258	8.592	4.528	2.519	2.067	1.284

Table 5.20 shows that the r^{th} moments decrease when α increases and θ decreases.

The central moments, the skewness, and the kurtosis are determined for different values of the parameters. The results are included in Table 5.21.

Table 5.21 The central moment, skewness and kurtosis at different values of parameters

	$\theta = 0.905$			$\theta = 0.779$			$\theta = 0.607$		
r	α			α			α		
	0.5	1	2	0.5	1	2	0.5	1	2
2	1.969	1.719	1.163	0.86	0.791	0.613	0.463	0.432	0.353
3	2.155	2.084	1.58	0.587	0.599	0.548	0.241	0.25	0.246
4	13.121	10.794	6.158	2.391	2.119	1.499	0.654	0.596	0.466
α_3	0.779	0.925	4.92	0.74	0.85	1.14	0.765	0.88	1.17
α_4	3.385	3.653	4.55	3.235	3.39	3.98	3.05	3.2	3.75

Table 5.21 represents that most central moments decrease when α increases and θ decreases. The distribution is positively skew for different values of α and θ. The distribution is leptokurtic for all selected values of parameters.

The mode values are determined at different values of the parameters. The results are included in Table 5.22.

Table 5.22 The mode at different values of parameters

$\theta = 0.905$			$\theta = 0.779$			$\theta = 0.607$		
α			α			α		
1.5	2.5	≥ 5	0.6	1	≥ 2	0.3	0.5	≥ 1
1	1	0	1	0	0	0	0	0

Table 5.22 represents the mode value when α increase and θ decreases. The mode does not change when α becomes large.

5.5.2 Performance of Estimators of DLFR Parameters

The parameters of the DLFR distribution are estimated by the proportion method, the moments method, and the maximum likelihood method.

The performance of estimators of the parameters α and θ are presented by using three methods of estimation such as bias, variance and mean squared error at selected values of parameters with different sample sizes in two cases.

Case I: known parameter $\alpha = 0.2$ and unknown parameter θ

The performance of the proportion estimator θ^*, the method-of-moments estimator $\tilde{\theta}$, and the maximum likelihood estimator $\hat{\theta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.23.

Table 5.23 Case I: known parameter $\alpha = 0.2$ and unknown parameter θ

		$\theta = 0.827$		
α_1	0.15	PM	MM	MLM
n		θ^*	$\tilde{\theta}$	$\hat{\theta}$
10	Bias Var MSE	-0.009 0.006 0.006	-0.014 0.004 0.004	-0.011 0.004 0.004
20	Bias Var MSE	-0.005 0.001 0.001	-0.012 0.003 0.003	-0.008 0.002 0.002
50	Bias Var MSE	-0.003 0.002 0.002	-0.002 0.001 0.001	-0.004 0.001 0.001
60	Bias Var MSE	-0.002 0.001 0.001	-0.001 0.00014 0.00014	-0.003 0.001 0.001
100	Bias Var MSE	-0.001 0.001 0.001	−0.00045 0.001 0.001	-0.001 0.001 0.001
200	Bias Var MSE	−0.00026 0.00025 0.00028	−0.0004 0.000045 0.000045	−0.00001 0.00011 0.00011

From Table 5.23 it can be observed that for the estimation of θ, the MM is the best method when the effective sample sizes are moderate or large. When the sample sizes become small (say, less than 30) the PM is the best method.

Case II: unknown parameters θ and α

The performance of the proportion estimators θ^* and α^*, the method-of-moments estimators $\tilde{\theta}$ and $\tilde{\alpha}$, and the maximum likelihood estimators $\hat{\theta}$ and $\hat{\alpha}$ are compared. The results are present in Table 5.24.

Table 5.24 Case II: unknown parameters θ and α

		$\theta = 0.607, \alpha = 0.3$					
α_1	0.15	PM		MM		MLM	
n		θ^*	α^*	$\tilde{\theta}$	$\tilde{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
10	Bias Var MSE	-0.097 0.072 0.081	140.312 1902000 1921000	-0.16 0.111 0.137	414.427 5546000 5717000	-0.091 0.057 0.065	5.387 1734 1763
20	Bias Var MSE	-0.078 0.062 0.068	417.386 5623000 5797000	-0.036 0.045 0.046	0.399 3.373 3.532	-0.057 0.056 0.06	4.676 1669 1691
50	Bias Var MSE	-0.035 0.017 0.018	0.091 0.727 0.736	-0.013 0.017 0.017	0.155 0.762 0.786	-0.012 0.01 0.01	0.114 0.529 0.542
60	Bias Var MSE	-0.011 0.008 0.008	0.051 0.352 0.355	-0.01 0.007 0.007	0.064 0.271 0.275	-0.007 0.007 0.007	0.101 0.326 0.336
100	Bias Var MSE	-0.006 0.004 0.004	0.041 0.151 0.152	-0.004 0.013 0.013	0.195 0.884 0.922	-0.007 0.003 0.004	0.032 0.127 0.128
200	Bias Var MSE	-0.004 0.002 0.002	0.009 0.066 0.066	-0.001 0.004 0.004	0.046 0.122 0.124	-0.004 0.004 0.004	0.02 0.112 0.112

Table 5.24 displays the results of estimating θ and α . In estimating θ , the MLM is better than the PM and the MM when the sample sizes are small (say, less than 30) or moderate. The PM is the best method only when the sample sizes become large. For the estimation of α, the MM is the appropriate method for small sample sizes (say, less than 30). When the sample sizes become moderate, the MLM is the best method. For large sample sizes, the PM and the ML method are the best.

5.6 The DMW (II) Distribution

The properties of the DMW (II) (α, β, θ) distribution are studied. The performance of the parameters by three methods of estimation at selected values of parameters with different sample sizes and different cases are discussed.

5.4.1 Properties of the DMW (II) Distribution

The properties of the DMW (II) (α, β, θ) distribution are studied. These properties include the r^{th} moments, the central moments, first and second ratio moments and mode at selected values of parameters.

The first four moments of the DMW (II) (α, β, θ) distribution are determined for different values of the parameters. The results are included in Table 5.25.

Table 5.25 The r^{th} moments at different values of parameters

	$\alpha = 0.1, \theta = 0.914$			$\alpha = 0.5, \theta = 0.914$			$\alpha = 1, \theta = 0.914$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	4.032	2.029	1.358	1.847	1.264	1.003	1.052	0.853	0.788
2	24.807	5.582	2.34	4.719	2.122	1.286	1.598	0.993	0.798
3	176.137	17.859	4.532	13.662	3.998	1.851	2.718	1.273	0.817
4	1362	63.795	9.609	43.228	8.227	2.982	5.037	1.832	0.856
	$\alpha = 0.1, \theta = 0.951$			$\alpha = 0.5, \theta = 0.951$			$\alpha = 1, \theta = 0.951$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	4.207	2.75	1.73	2.472	1.637	1.26	1.4	1.101	0.925
2	29.13	9.809	3.654	7.811	3.347	1.944	2.552	1.558	1.029
3	225.084	40.25	8.627	27.519	7.614	3.326	5.151	2.472	1.237
4	1854	182.845	22.147	104.644	18.755	6.131	11.236	4.303	1.654
	$\alpha = 0.1, \theta = 0.896$			$\alpha = 0.5, \theta = 0.896$			$\alpha = 1, \theta = 0.896$		
r	β			β			β		
	1	2	3	1	2	3	1	2	3
1	3.741	1.816	1.245	1.652	1.148	0.926	0.94	0.78	0.743
2	21.631	4.564	2.001	3.905	1.8	1.108	1.339	0.858	0.746
3	146.602	13.372	3.621	10.521	3.176	1.474	2.144	1.013	0.752
4	1094	43.878	7.19	31.127	6.139	2.206	3.779	1.322	0.764

Table 5.25 Shows that the r^{th} moments decrease when α, β increase and θ fixed.

The central moments, skewness, and kurtosis of the DMW (II) Distribution are determined at different values of the parameters. The results are included in Table 5.26.

Table 5.26 The central moment, skewness and kurtosis at different values of parameters

	$\alpha = 0.1, \theta = 0.951$			$\alpha = 0.5, \theta = 0.951$			$\alpha = 1, \theta = 0.951$		
r	β 1	2	3	β 1	2	3	β 1	2	3
2	8.037	2.244	0.662	1.703	0.666	0.356	0.592	0.345	0.173
3	20.628	0.931	0.015	-0.204	-0.048	-0.021	-0.079	-0.003	-0.036
4	160.039	13.538	1.204	6.94	1.167	0.324	0.877	0.337	0.163
α_3	0.905	0.277	0.027	-0.091	-0.088	-0.099	-0.173	-0.014	-0.5
α_4	2.477	2.688	2.748	2.393	2.634	2.571	2.505	2.832	5.62
	$\alpha = 0.1, \theta = 0.914$			$\alpha = 0.5, \theta = 0.914$			$\alpha = 1, \theta = 0.914$		
r	β 1	2	3	β 1	2	3	β 1	2	3
2	7.722	1.467	0.495	1.306	0.523	0.279	0.492	0.265	0.177
3	10.511	0.582	0.009	0.117	-0.009	0.0006	0.001	-0.027	-0.09
4	134.842	5.91	0.675	3.965	0.699	0.279	0.539	0.236	0.096
α_3	0.49	0.328	0.025	0.078	-0.023	0.004	0.002	-0.198	-1.216
α_4	2.261	2.746	2.755	2.325	2.56	3.623	2.23	3.371	3.096
	$\alpha = 0.1, \theta = 0.896$			$\alpha = 0.5, \theta = 0.896$			$\alpha = 1, \theta = 0.896$		
r	β 1	2	3	β 1	2	3	β 1	2	3
2	7.27	1.264	0.45	1.177	0.482	0.252	0.456	0.249	0.194
3	9.921	0.487	0.008	0.182	0.001	-0.018	0.03	-0.045	-0.09
4	123.839	4.416	0.556	3.211	0.581	0.244	0.472	0.183	0.086
α_3	0.506	0.34	0.026	0.143	0.002	-0.142	0.097	-0.362	-1.058
α_4	2.343	2.765	2.738	2.318	2.504	3.873	2.28	2.951	2.324

Table 5.26 represents that most of central moments decreases when β increase and the values of α increase. For different values of β and θ, the distribution

becomes positively skewed when α small, otherwise it becomes negatively skewed when α large. The distribution is platykurtic for most values of parameters and it becomes leptokurtic when all parameters become large.

The mode of the DMW (II) Distribution is determined at different values of the parameters. The results are included in Table 5.27.

Table 5.27 The mode at different values of parameters

$\alpha = 0.1, \theta = 0.951$				$\alpha = 0.5, \theta = 0.951$				$\alpha = 1, \theta = 0.951$			
β				β				β			
1	2	3	≥ 4	1	2	3	≥ 4	1	2	3	≥ 4
6	3	2	1	3	2	1	1	2	1	1	1
$\alpha = 0.1, \theta = 0.914$				$\alpha = 0.5, \theta = 0.914$				$\alpha = 1, \theta = 0.914$			
β				β				β			
1	2	3	≥ 4	1	2	3	≥ 4	1	2	3	≥ 4
3	2	1	1	2	1	1	1	1	1	1	1
$\alpha = 0.1, \theta = 0.896$				$\alpha = 0.5, \theta = 0.896$				$\alpha = 1, \theta = 0.896$			
β				β				β			
1	2	3	≥ 4	1	2	3	≥ 4	1	2	3	≥ 4
2	2	1	1	2	1	1	1	1	1	1	1

Table 5.27 shows that the mode values become small when α and β increase. The mode does not changes when β ≥ 4.

5.6.2 Performance of Estimators of DMW (II) Parameters

The performance of estimators of the parameters *α, β and θ* are presented by using three methods of estimation such as bias, variance and mean squared error at selected values of parameters with different sample size in four cases.

Case I: known parameters $\boldsymbol{\alpha = 0.15, \beta = 2}$ and unknown parameter $\boldsymbol{\theta}$

The performance of the proportion estimator θ^*, the method-of-moments estimator $\tilde{\theta}$, and the maximum likelihood estimator $\hat{\theta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.28.

Table 5.28 Case I: known parameters $\alpha = 0.15, \beta = 2$ and unknown parameter θ

		$\theta = 0.819$		
λ	0.2	PM	MM	MLM
n		θ^*	$\tilde{\theta}$	$\hat{\theta}$
10	Bias Var MSE	-0.005 0.003 0.003	-0.021 0.004 0.005	-0.016 0.002 0.003
20	Bias Var MSE	-0.003 0.006 0.006	-0.007 0.002 0.002	-0.008 0.001 0.001
50	Bias Var MSE	-0.003 0.001 0.001	-0.003 0.001 0.001	-0.003 0.001 0.001
60	Bias Var MSE	-0.002 0.003 0.003	-0.002 0.001 0.001	-0.001 0.001 0.001
100	Bias Var MSE	-0.001 0.002 0.002	-0.00026 0.00013 0.00013	-0.00037 0.00010 0.00010
200	Bias Var MSE	-0.00036 0.001 0.001	-0.00022 0.00025 0.00025	0.00026 0.000039 0.000039

From Table 5.28 it can be observed that for the estimation of θ, the MLM is always better than the PM and the MM for different sample sizes.

Case II: known parameter $\beta = 2$ and unknown parameters θ *and* α

The performance of the proportion estimators θ^* *and* α^*, the method-of-moments estimators $\tilde{\theta}$ *and* $\tilde{\alpha}$, and the maximum likelihood estimators $\hat{\theta}$ *and* $\hat{\alpha}$ are compared. The results are present in Table 5.29.

Table 5.29 Case II: known parameter $\boldsymbol{\beta = 2}$ and unknown parameters $\boldsymbol{\theta, \alpha}$

		$\theta = 0.819, \alpha = 0.15$					
λ	0.2	PM		MM		MLM	
n		θ^*	α^*	$\tilde{\theta}$	$\tilde{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
10	Bias Var MSE	-0.05 0.049 0.051	0.53 3.291 3.572	-0.007 0.011 0.011	0.077 0.062 0.068	-0.029 0.006 0.006	0.016 0.086 0.086
20	Bias Var MSE	-0.021 0.021 0.022	0.129 0.435 0.451	-0.005 0.009 0.009	0.073 0.072 0.077	-0.02 0.002 0.003	-0.028 0.007 0.008
50	Bias Var MSE	-0.008 0.008 0.008	0.019 0.056 0.056	-0.002 0.005 0.005	0.03 0.027 0.028	-0.01 0.003 0.003	0.003 0.009 0.009
60	Bias Var MSE	-0.007 0.006 0.006	0.015 0.069 0.069	-0.001 0.004 0.004	0.026 0.023 0.023	-0.005 0.004 0.004	0.009 0.019 0.019
100	Bias Var MSE	0.001 0.002 0.002	0.001 0.01 0.01	0.00036 0.002 0.002	0.005 0.005 0.005	-0.001 0.001 0.001	0.004 0.015 0.015
200	Bias Var MSE	0.000062 0.001 0.001	0.001 0.01 0.01	0.000031 0.00036 0.00036	0.00014 0.002 0.002	0.00015 0.003 0.003	−0.00022 0.001 0.001

Table 5.29 represents the results for the estimation of θ and α when $\beta = 2$. known. For the estimation of θ, the MLM is better than the PM and the MM when the sample sizes are small (say, less than 30). When the sample sizes become moderate or large the MM is the best. For the estimation of α, the MLM method is the best for different sample sizes.

Case III: known parameter $\boldsymbol{\alpha = 0.15}$ and unknown parameters $\boldsymbol{\theta}$ $\boldsymbol{and}$ $\boldsymbol{\beta}$

The performance of the proportion estimators θ^* and β^*, the method-of-moments estimators $\tilde{\theta}$ and $\tilde{\beta}$, and the maximum likelihood estimators $\hat{\theta}$ and $\hat{\beta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.30.

Table 5.30 Case III: known parameter $\boldsymbol{\alpha = 0.15}$ and unknown parameters $\boldsymbol{\theta, \beta}$

		$\theta = 0.819, \beta = 2$					
λ	0.2	PM		MM		MLM	
n		θ^*	β^*	$\tilde{\theta}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\beta}$
10	Bias Var MSE	-0.007 0.015 0.015	1.608 20.785 23.371	0.011 0.001 0.001	0.105 0.051 0.062	-0.131 0.203 0.22	-0.823 9.89 10.568
20	Bias Var MSE	-0.006 0.012 0.012	1.208 16.08 17.54	0.008 0.005 0005	0.22 0.382 0.43	-0.128 0.204 0.221	-0.572 11.062 11.389
50	Bias Var MSE	-0.002 0.005 0.005	0.106 0.463 0.474	0.003 0.002 0.002	0.081 0.097 0.103	-0.081 0.529 0.535	0.823 3.603 4.28
60	Bias Var MSE	-0.002 0.003 0.003	0.071 0.284 0.289	0.001 0.002 0.002	0.085 0.084 0.092	-0.031 0.155 0.156	-0.015 7.756 7.756
100	Bias Var MSE	0.001 0.001 0.001	0.01 0.049 0.049	0.00015 0.002 0.002	0.041 0.073 0.074	-0.002 0.008 0.008	0.211 0.359 0.404
200	Bias Var MSE	0.00021 0.001 0.001	0.007 0.05 0.05	0.000027 0.001 0.001	0.022 0.051 0.052	-0.001 0.004 0.004	0.089 0.161 0.169

Table 5.30 Shows the results for the estimation of θ *and* β when $\alpha = 0.15$. For the estimation of θ, the MM is the appropriate method for all sample sizes. For the estimation of β, the MM is the best when the effective sample sizes are small (say, less than 30) or moderate. The PM is the best when the sample sizes are large.

Case IV: unknown parameters $\boldsymbol{\theta, \alpha}$ *and* $\boldsymbol{\beta}$

The performance of the proportion estimators θ^*, α^* *and* β^*, the method-of-moments estimators $\tilde{\theta}$, $\tilde{\alpha}$ *and* $\tilde{\beta}$, and the maximum likelihood estimators $\hat{\theta}$, $\hat{\alpha}$ *and* $\hat{\beta}$ are compared. The comparison included the bias, variance, and mean squared error. The results are present in Table 5.31.

Table 5.31 Case IV: unknown parameters θ, α and β

		$\theta = 0.819, \alpha = 0.15, \beta = 2$								
λ	0.2	PM			MM			MLM		
n		θ^*	α^*	β^*	$\tilde{\theta}$	$\tilde{\alpha}$	$\tilde{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$
10	Bias	-.211	0.794	-.882	0.078	0.7	-0.853	0.168	-0.035	1.821
	Var	2.922	8.968	26.593	0.008	0.625	1.467	0.00027	1.876	16.002
	MSE	2.966	9.598	27.37	0.014	1.115	2.194	0.028	1.877	19.318
20	Bias	-.037	0.636	-.577	0.039	0.482	-0.635	0.16	-0.002	1.805
	Var	0.26	13.459	28.532	0.009	0.659	1.718	0.001	0.000012	1.068
	MSE	0.261	13.864	28.866	0.01	0.892	2.121	0.026	0.000018	4.325
50	Bias	-.005	0.195	-.246	0.012	0.172	-0.243	0.026	0.153	-0.259
	Var	0.014	1.084	3.201	0.005	0.265	0.9	0.001	0.103	0.443
	MSE	0.014	1.122	3.262	0.006	0.295	0.959	0.002	0.127	0.51
60	Bias	-.005	0.103	-.119	0.01	0.132	-0.192	0.014	0.277	-0.506
	Var	0.007	0.544	1.801	0.004	0.189	0.721	0.007	0.431	1.231
	MSE	0.007	0.555	1.816	0.004	0.206	0.758	0.007	0.508	1.488
100	Bias	-.002	0.025	-.017	0.001	0.004	0.014	0.007	0.017	0.136
	Var	0.003	0.135	1.345	0.001	0.034	0.102	0.003	0.336	0.825
	MSE	0.003	0.135	1.345	0.001	0.034	0.103	0.003	0.337	0.844
200	Bias	−.00045	.0000003	.0001	0.0003	0.001	0.003	0.002	-0.003	0.065
	Var	0.004	0.013	0.497	0.001	0.027	0.119	0.002	0.056	0.169
	MSE	0.004	0.013	0.497	0.001	0.027	0.119	0.002	0.056	0.173

From Table 5.31 it is observed that for the estimation of θ, the MM is always better than the PM and the MLM for different sample sizes. For the estimation of α, the MLM is the best when the effective sample sizes are small (say, less than 30) or moderate. When the effective sample sizes are large the MM is the best one to be applied. For the estimation of β, the MM is always the best method for all sample sizes.

5.7 Summary

A MathCAD simulation is conducted to study the properties and the performance of all estimators for the new discretized models. Many points of interest can be listed as:

1. The r^{th} moments decrease when β increases for all studied discretized distributions.
2. The mode value starts to be fixed when β increases for all discretized distributions.
3. Evaluation of the central moments shows that
   (a) For small values of β, the DMWE and DChen distributions become positively skewed and platykurtic. These distributions become negatively skewed and leptokurtic when β large.
   (b) DMW (I) and DLFR distributions become positively skew. These distributions sometimes become leptokurtic or platykurtic depends on β small or large, respectively.
   (c) For large values of β, The DMW (II) distribution becomes negatively skewed when α and θ become large and positively skewed when α and θ become small. For the kurtosis the distribution becomes leptokurtic when all parameters increase and become platykurtic when some parameters decrease.
4. For the DMWE and DChen distributions, the PM is better when the sample sizes are small. For moderate or large sample sizes, the MM is the best one.
5. For the DMW (I) and DLFR distributions, the PM and the MM are commonly better when the sample sizes are small, moderate or large.
6. For the DMW (II) distribution, the MM and MLM are good for different sample sizes.

Chapter VI

Conclusion and Recommendations

This chapter involves the summary of thesis, its results, and some suggestions of future work.

6.1 Introduction

In this thesis, Chapter I presents an introduction and comprehensive information about creating a general class of discretized continuous distributions as well as, the distributions that were applied in this thesis. Chapter II describes the definitions and notations that used in the thesis.

Chapter III presents the historical review of the discretizing methods and a review about the distributions used in the thesis. Chapter IV investigates the theoretical statistics of the distributions such as the properties and methods of estimation. Finally, Chapter V uses the MathCAD software, to study some properties and methods of estimations of the parameters.

6.2 Conclusion

The purpose of this thesis is to explore three new lifetime distributions suitable and flexible for modeling discrete data by presenting a general class of discretized continuous distribution. The new models are DMWE, DMW (I), and DMW (II) that contain three parameters α, β, θ. We carry out a theoretical study of the obtained distributions, discussing their distributional properties, developing the measures of reliability in the discrete case and describing the distributions graphically. However, the closed form for the distributional properties could not achieve. In addition, the parameter estimation of the discussed distributions are obtained by using three methods of estimation PM, MM, and MLE. For each method four cases are studied:

Case I: unknown parameter θ and known parameters $and\ \beta$.

Case II: unknown parameters $\theta\ and\ \alpha$ and known parameter β.

Case III: unknown parameters $\theta\ and\ \beta$ and known parameter α .

Case IV: unknown parameters $\theta, \alpha\ and\ \beta$.

The estimators cannot be obtained in exact form, except case I for PM only can derive the parameter θ as exact solution. Therefore, simulation studies are required to achieve the parameter estimation.

Special cases from DMWE and DMWI are obtained which are called DChen and DLFR distributions respectively. The distributional properties are discussed, and the parameters are estimated.

Simulation studies are developed to investigate the parameters of the DMWE, DMWI and DMWII distributions and for the special cases DChen and DLFR distributions. The results of the three methods are compared. The r^{th} moments, the central moments, the skewness, the kurtosis and the mode at different values of parameters are computed.

Simulation results of the parameter estimation on the DMWE showed that the performance of the estimators of θ is better by using the MM when the sample sizes become moderate and the MLM is the best method to apply when the sample size becomes large. For the performance of the estimator α, the PM and MLM are better when the sample sizes are small or moderate. When the sample sizes become large then the MM is the best one. For the performance of the estimators of β the MM is the appropriate method at different sample sizes.

For DChen distribution it appears that the performance estimators of θ is better by using the PM when the sample sizes become small. When the sample sizes are moderate or large the MM is the best method. For the performance estimator of β the MM is the appropriate method at different sample sizes.

For DMW (I) distribution it appears that the performance of estimators of θ is better by using the PM or MM for all sample sizes. For the performance of estimators of α, the MM is the appropriate method when the sample sizes are small or moderate but the PM is the best one to apply when the sample size becomes large. For the performance estimator of β the MM is the appropriate method at different sample sizes.

For the DLFR it appears that the performance of estimators of θ is better by using the PM when the sample sizes become small. When the sample sizes are large the MM is the best method. When the sample sizes are moderate then the MLM is the best. For the performance of estimators of α the MM is the best method when the sample sizes become small. The MLM is the best when the sample sizes become moderate. When the sample sizes become large then the PM is the best.

For the DMW (II) it appears that the performance of estimators of θ, the MM and MLM are the appropriate methods at different sample sizes. For the performance of estimators of α, the MLM is the appropriate method when the sample sizes are small or moderate and the MM is the best one to apply when sample sizes are large. For the performance of estimators of β the MM is the appropriate method at different sample sizes.

6.3 Future Recommended work

Based on our present study, there are some suggestions for future researches regarding the following points:

1. Creating new discretized distributions, using the general class and studying their properties, estimating parameters and related results.
2. Estimating the parameters of the obtained distributions in Bayesian and comparing the results with our results.
3. Constructing confidence intervals for the discretized distributions.
4. Estimating the parameters based on censored samples.
5. Comparing the distribution results in the continuous and discrete cases.

LIST OF REFERENCES

Alwasel, I. (2009). Statistical inference of a competing risks model with modified Weibull distributions, International Journal of Mathematical Analysis, Vol. 3, No. 19: 905-918.

Al-Dayian, G. and Al-Huniti, A. (2012). Discrete Burr type III distribution, American Journal of Mathematics and Statistics, Vol. 2, No. 5: 145-152.

Bain, L. and Engelhrdt, M. (1992). Introduction to Probability and Mathematical Statistics, Duxbury Classic Series.

Barlow, R. (2001). System and Bayesian reliability, World Scientific Publishing Co. Pte-ltd.

Chen, Z. (2000). A new two - parameter lifetime distribution with bathtub shape or increasing failure rate function, Statistics and Probability Letters, Vol. 49, No. 2: 155-161.

Deniz, G. and Ojeda, C. (2011). The discrete Lindley distribution: Properties and Applications, Journal of Statistics: Computation and Simulation, Vol.81, No. 11: 1405-1416.

Gasmi, S. and Berzig, M. (2011). Parameters estimation of the modified Weibull distribution based on type I censored sample, Applied Sciences, Vol. 5, No. 59: 2899-2917.

Inusah, S. and Kozubowski, T. (2006). A discrete analogue of the Laplace distribution, Journal of Statistical Planning and Inference, Vol. 136, No. 3: 1090-1102

Khan, M., Khalique, A. and Abouammoh, A. (1989). On Estimating parameters in a discrete Weibull distribution, IEEE Transactions on Reliability, Vol. 38, No. 3: 348-350.

Kemp, A. (1997). Characterizations of a discrete normal distribution, Journal of Statistical Planning and Inference, Vol. 63, No. 2: 223-229.

Krishna, H. and Pundir, P. (2009). Discrete Burr and discrete Pareto distributions, Statistical Methodology, Vol. 6, No. 2: 177-188.

Lai, C., Xie, M. and Murthy, D. (2003). A modified Weibull distribution, IEEE Transactions on Reliability, Vol. 52, No. 1: 33-37.

Lai, M. and Alamatsaz, M. (2010). A discrete inverse Weibull distribution and estimation of its parameters, Statistical Methodology, Vol.7, No. 2: 121-132.

Mario, A., Wiliam, W., Roberto, G., and Yulia, U. (2008). An Introduction to Survival Analysis Using Stata, Texas: Stata Press.

Mood, A., Graybill, F. and Boes, D. (1974). Introduction to the Theory of Statistics, New York: McGraw-Hill.

Nakagawa, T. and Osaki, S. (1975). The discrete Weibull distribution, IEEE Transactions on Reliability, Vol. R-24, No. 5: 300-301.

Nakagawa, T. (1978). Discrete extreme distribution, IEEE Transactions on Reliability, Vol. R-27, No. 5: 367-368.

Nadarajah, S. (2005). On the moments of the modified Weibull distribution, Reliability Engineering and System Safety, Vol.90, No. 1: 114-117.

Ng, H. (2005). Parameter estimation for a modified Weibull distribution, for progressively type-II censored samples, IEEE Transactions on Reliability, Vol.54, No. 3: 374-380.

Perdona, G., Dms-Fmrp, Faepa-Rp and Louzada-Neto, F. (2008). Interval estimation for the parameters of the modified Weibull distribution model with censored data: a simulation study, TEMA-Tendências em Matemática Aplicada e Computacional, Vol. 9, No. 3: 437-446.

Roy, D. (2004). Discrete Rayleigh distribution, IEEE Transactions on Reliability, Vol. 53, No. 2: 255-260.

Stein, W. and Dattero, R. (1984). A new discrete Weibull distribution, IEEE Transactions on Reliability, Vol. R-33, No. 2: 196-197.

Sarhan, A. and Zaindin, M. (2009a). Modified Weibull distribution, Applied Sciences, Vol. 11, pp. 123 - 136.

Sarhan, A. and Zaindin, M. (2009b). Parameters estimation of the modified Weibull distribution, Applied Sciences, Vol. 3, No. 11:541-550.

Xie, M., Tang, Y., and Goh, T. N. (2002). A modified Weibull extension with bathtub-shaped failure rate function, Reliability Engineering and System Safety, Vol. 76, No. 3: 279-85.

Zaindin, M. (2010). Parameter estimation of the modified Weibull model based on grouped and censored data, Applied Sciences, Vol. 10, No. 2: 122-132.

VITA

- Taghreed Abdulrahman Al-Masoud.
- Born in Jeddah, Saudi Arabia on October 13, 1985.
- Received Bachelor of Science (B. S.) degree in Statistics from King Abdulaziz University (KAU), Jeddah, Saudi Arabia in 2008.
- E-mail: taalmasoud@yahoo.com

التوزيع المنفصل لفصل من التوزيعات الاحتمالية المتصلة

تغريد عبد الرحمن حامد المسعود

المستخلص

لقد بات من الضروري جدا الاهتمام بجودة وصلاحية المنتجات بكافة أنواعها حتى تتمكن من الصمود والمنافسة في الأسواق نظرا لتعدد أشكال ومنابع المنتج الواحد. فمن المعروف أن المستهلك تهمه جودة المنتجات الصناعية بكافه أنواعها التي تعرض في الأسواق وان تكون ذات كفاءة عالية وعمرا أطول. وانسجاما مع متطلبات المستهلك فان أصحاب المصانع في بحث دءوب لجذب المستهلك إلى منتجات مصانعهم.

ومن أهم ما نتج عن هذه الأبحاث هو ظهور ما يسمى بشهادات الضمان والتي تعطي رواجا وأفضلية لدى المستهلك عن المنتج البديل والخالي من الضمان. وعليه فمن الضروري جدا تسليط الضوء على كيفية تحديد المدة المناسبة لشهادات الضمان بدقة، وإلا فان الخطأ في تحديدها يمكن أن يكبد الشركات خسائر كبيره. إن تحديد المدة المناسبة لمثل هذه الشهادات يستلزم جمع المعلومات عن المنتج عن طريق تصميم ما يسمى تجارب اختبارات الحياة life testing experiments أو تجارب اختبارات الصلاحية Reliability tests وذلك قبل عرض المنتج بالأسواق. أن المعلومات المتوقع الحصول عليها من مثل هذه التجارب- بجانب أهميتها في تحديد مدة شهادات الضمان- يمكن استخدامها في جوانب اخرى. فعلى سبيل المثال في الدراسات الصيدلانية نرغب في تصميم تجارب اختبارات عمر المنتج للعقاقير لتحديد مدة فعالية هذا العقار وتاريخ صلاحيته، وهناك الكثير من المجالات الأخرى التي يكون فيها تصميم مثل هذه التجارب في غاية الأهمية.

في تجارب اختبارات الحياة، أحيانا يكون قياس مدة صلاحية المنتج أو طول بقائه مستحيل قياسها بالمقياس المتصل. مثلا عند تشغيل الجهاز وإيقافه وقت الحياة في التشغيل والإيقاف هو متغير عشوائي منفصل. في بعض الحالات بشكل خاص، صلاحية البيانات تقاس بعدد مرات التشغيل التي من الممكن أن يتحملها الجهاز قبل أن يتوقف عن العمل. وعند تحليل مدة البقاء من الممكن أن تسجل عدد الأيام التي من الممكن أن يعيشها مريض سرطان الرئة بعد تلقيه العلاج.

في هذا السياق، التوزيع الهندسي Geometric distribution وتوزيع ذي الحدين السالب Negative binomial distribution هما توزيعان منفصلان بديلان للتوزيع الأسى Exponential distribution وتوزيع جاما Gamma distribution.

في هذه الرسالة تناولنا فصل من التوزيعات الاحتمالية المتصلة وقمنا بتحويله الى فصل من التوزيعات الاحتمالية المنفصلة. من فصل التوزيعات الاحتمالية المتصلة اخترنا التوزيعات التالية: توزيع وايبل المعدل الممدد modified Weibull extended distribution وتوزيع وايبل المعدل من النوع الأول modified Weibull Type I وتوزيع وايبل المعدل من النوع الثاني modified Weibull Type II.

تم الحصول على الصيغ المنفصلة لهذه التوزيعات وهى على الترتيب: توزيع وايبل المعدل الممدد المنفصل discrete modified Weibull extended distribution ، وتوزيع وايبل

وتوزيع وايبل المنفصل المعدل من النوع الأول discrete modified Weibull type I وتوزيع وايبل المنفصل المعدل من النوع الثاني discrete modified Weibull type II.

تم تناول التوزيعات المنفصلة المشار إليها بدراسة خصائصها الإحصائية. تم استنباط بعض التوزيعات الاحتمالية المنفصلة كحالات خاصة من التوزيعات المشار إليها. من التوزيعات الاحتمالية المنفصلة التي تم تناولها نذكر: توزيع معدل الفشل الخطى المنفصل discrete linear failure rate distribution ، توزيع تشن المنفصل discrete Chen distribution . قمنا كذلك بإجراء دراسة محاكاة للتوزيعات الاحتمالية المنفصلة التي تم تناولها.

وتتكون الرسالة من ستة أبواب. خصصنا الباب الأول للتقديم لموضوع البحث الوارد بالرسالة. أما الباب الثاني فقد تم تخصيصه لعرض المفاهيم والتعاريف ذات الصلة بتوزيعات الحياة. وقد خصص الباب الثالث لعرض المسح التاريخي والنتائج العلمية المتعلقة بموضوع الرسالة. أما الباب الرابع فقد خصص لعرض ومناقشة النتائج العلمية الجديدة التي حصلنا عليها. الباب الخامس تم تخصيصه لعرض نتائج دراسة المحاكاة التي تم دراستها. الخاتمة لنتائج البحث والتوصيات للبحث المستقبلي تم عرضهما في الباب السادس. وتحتوى الرسالة كذلك على قائمة لكل من: محتوى الرسالة ، الرموز المستخدمة ، الأشكال ، الجداول ، وأخيرا قائمة بالمراجع المستخدمة.

التوزيع المنفصل لفصل من التوزيعات الاحتمالية المتصلة

إعداد

تغريد عبدالرحمن المسعود

بحث مقدم لنيل درجة الماجستير في العلوم(إحصاء)

إشراف

د/ سهيلة حامد الجيار

د/ محمود عبد المؤمن عطاالله

كلية العلوم

جامعة الملك عبدالعزيز

جدة - المملكة العربية السعودية

جمادى الثانى ١٤٣٤ هـ - مايو ٢٠١٣ م